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THE GAUSS TEST FOR RELATIVE CONVERGENCE.*

By R. L. GOODSTEIN.

Our problem is to establish in extended *recursive number theory* a test for *relative convergence* analogous to the familiar Gauss test for positive series. Recursive number theory is a formal system admitting only (primitive or multiply) recursive functions or relations (and therefore only *bounded* existential and universal operators). Formulations of recursive number theory have been given, *inter alia*, by [1], [2] and [3]¹; we consider a simple extension of recursive number theory to rational functions of positive integral arguments. Propositions and proofs of the extended system R may be translated into the parent system with the help of the familiar definitions of fractions in terms of pairs of natural numbers and of rational numbers in terms of pairs of fractions. A *fractional recursive function* $f(n_1, n_2, \dots, n_k)$ is a pair of recursive functions $a(n_1, n_2, \dots, n_k)/b(n_1, n_2, \dots, n_k)$ such that $b(n_1, n_2, \dots, n_k) > 0$ for all natural numbers n_1, n_2, \dots, n_k and $a/b \cong a'/b'$ according as $ab' \cong a'b$, and $a/b + a'/b' = (ab' + a'b)/bb'$, $(a/b) \cdot (a'/b') = aa'/bb'$, and a *rational recursive function* $r(n_1, n_2, \dots, n_k)$ is a pair of fractional recursive functions $[f(n_1, n_2, \dots, n_k), g(n_1, n_2, \dots, n_k)]$ such that $[f, g] \cong [f', g']$ according as $f + g' \cong f' + g$ and $[f, g] + [f', g'] = [f + f', g + g']$, $[f, g] \cdot [f', g'] = [ff' + gg', fg' + f'g]$, etc.

The concept of *relative convergence* was introduced in [2] to serve for the expression in a rational field of an analogue of a convergent sequence of real numbers. A rational recursive function $f(m, n)$ is said to be convergent in m relative to n if (roughly speaking) $f(m, n)$ satisfies the Cauchy criterion for convergence in m only for sufficiently great values of n ; convergence relative to n does not involve convergence for any fixed value of n , since for instance m/n is convergent in m relative to n , but m/n is not convergent for any given value of n .

Definitions in R. A rational recursive function $s(n)$ is said to be convergent if it is associated with a recursive function $n(k)$ such that $\{n \geq n(k)\} \rightarrow |s(n) - s(n(k))| < 1/(k+1)$, where n, k are free variables. A rational number p/q is said to be the limit of $s(n)$ (and $s(n)$ is said to tend to p/q ,

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¹ The numbers in square brackets refer to the bibliography at the end of the paper.

denoted by $s(n) \rightarrow p/q$ if $s(n)$ is associated with a recursive function $n(k)$ such that $\{n \geq n(k)\} \rightarrow |p/q - s(n)| < 1/(k+1)$, where n, k are free variables. If $s(n) \rightarrow p/q$ then $s(n)$ is convergent, for $|s(n) - s(n(k))| \leq |s(n) - p/q| + |s(n(k)) - p/q| < 2/(k+1)$, when $n \geq n(k)$. The function $s(n)$ is *divergent* if it is associated with a recursive $d(n)$ and a constant $k \geq 1$ such that $|s(n + d(n)) - s(n)| \geq 1/k$, where n is a free variable.

The function $s(n)$ is said *not* to tend to a limit if there are recursive functions $\nu(p, q) \geq 1$, $n(p, q, r) \geq r$, such that

$$\{n = n(p, q, r)\} \rightarrow |s(n) - p/(q+1)| \geq 1/\nu(p, q),$$

where p, q and r are free variables.

In a rational field, of course, a convergent function does not necessarily tend to a limit. For instance, if $s(n) = u(n)/n!$, where $u(0) = 1$, $u(n+1) = (n+1)u(n) + 1$ then $s(n)$ converges, for if $n > q$ then $0 < s(n) = s(q) < 1/q(q!)$; but

$$\begin{aligned} \{s(q) - p/q\}(q!) &= u(q) - p\{(q-1)!\} \\ &= (q-1)[q \cdot u(q-2) + 1 - p\{(q-2)!\}] + 2 \end{aligned}$$

$= 2(\text{mod } (q-1))$, $q \geq 4$. Hence $|s(n) - p/q| \geq (1 - 1/q)/q!$, for $n > q \geq 4$; in the cases $q = 1, 2, 3$, for $n > 4$ the difference $|s(n) - p/q|$ is least for $p = 8$, $q = 3$ and $|s(n) - 8/3| > 1/24$. Thus $s(n)$ does not tend to p/q for any p, q .

The restriction to bounded operators does not seriously complicate the theory of ordinary convergence. The principal classical result lost by this restriction is that which asserts the convergence of a bounded monotonic sequence. For a proof in R that any non-decreasing recursive function $f(n)$ is associated with a recursive $n_f(k)$ such that

$$n \geq n_f(k) \rightarrow |f(n) - f(n_f(k))| < 1/(k+1)$$

entails that an equivalent of the formula

$$(n)(k)\{n \geq n_f(k) \rightarrow |f(n) - f(n_f(k))| < 1/(k+1)\}$$

is provable in the system Z_μ of Hilbert-Bernays [3], and so if $T(m)$ is a recursive relation such that $T(v)$ holds for any assigned numeral v but $(m)T(m)$ is non-demonstrable in Z_μ , and if $f(n)$ is the characteristic function of the recursive relation $(m)\{m \leq n \& T(m)\}$, then $f(n)$ is demonstrably recursive, non-decreasing and bounded by unity, in Z_μ . By hypothesis there

is a recursive function $n_f(x)$ in R and so a recursive term $n_f(x)$ in Z_μ such that the formula

$$(n)\{n \geq n_f(0) \rightarrow |f(n) - f(n_f(0))| < 1\}$$

is demonstrable in Z_μ , whence it follows that

$$(n)\{n \geq n_f(0) \rightarrow f(n) = f(n_f(0))\}$$

is demonstrable. Since $n_f(x)^*$ is primitive recursive, there is a numeral w such that the equation $n_f(0) = w$ is provable in Z_μ ; but $T(r)$ is provable in Z_μ for $r = 1, 2, \dots, w$, so that $(n)(n \leq w \& f(n) = 0)$ is provable, and in particular $f(n_f(0)) = 0$ is provable. It follows that $(n)(f(n) = 0)$ is demonstrable in Z_μ , from which we conclude that $(m)T(m)$ is demonstrable in Z_μ , in contradiction to the hypothesis on $T(m)$.

We preface our discussion of relative convergence by a brief examination of the validity in R of the elementary tests for ordinary convergence of positive series. We denote by $a(n)$, $u(n)$, etc., positive (non-zero) recursive functions, and define for any function $f(n)$,

$$\sum_{r=m}^{m+n} f(r) = f(m), \quad \sum_{r=m}^{m+n+1} f(r) = \sum_{r=m}^{m+n} f(r) + f(m+n+1),$$

so that, e. g., $\sum_{r=m}^{m+n} a(r)$ is monotonic increasing with n .

If x is a rational variable then $\sum_{r=0}^n x^r$ is convergent for $0 < x < 1$, and divergent for $x \geq 1$; for if $x \neq 1$, $\sum_{r=0}^{n-1} x^r = (1 - x^n)/(1 - x)$ (proof by induction), and if $y > 0$, $(1 + y)^n \geq 1 + ny$ (proof by induction), so that $1/(1 + y)^n \rightarrow 0$, whence, taking $y = (1/x) - 1 > 0$, $\sum_{r=0}^{n-1} x^r \rightarrow 1/(1 - x)$, provided that $0 < x < 1$. If $x \geq 1$ then $\sum_{r=0}^{n+1} x^r - \sum_{r=0}^n x^r = x^{n+1} \geq 1$, so that $\sum_{r=0}^n x^r$ diverges. Similarly if $a(n)$ does not tend to zero then $\sum_{r=0}^n a(r)$ diverges, for $\sum_{r=0}^{n+1} a(r) - \sum_{r=0}^n a(r) = a(n+1) \geq 1/\nu(0, 0)$ provided that $n+1 = n(0, 0, p)$.

If $\sum_{r=0}^n a(r)$ diverges then $\{1/\sum_{r=m}^n a(r)\} \rightarrow 0$ for any fixed m ; let $s(n) = \sum_{r=m}^{m+n} a(r)$ and $\phi(0) = 0$, $\phi(n+1) = \phi(n) + d(\phi(n))$, so that $s(\phi(n+1)) - s(\phi(n)) \geq 1/k$, by hypothesis, whence $s(\phi(n)) \geq n/k$ and so, for $p \geq \phi(n)$, $1/s(p) \leq 1/s(\phi(n)) \leq k/n \rightarrow 0$.

The elementary tests.

The comparison tests. (i) If $u(n), v(n)$ are associated with a non-zero constant k , such that $u(n) \leq kv(n)$, then the convergence of $\sum_{r=0}^n v(r)$ entails that of $\sum_{r=0}^n u(r)$, and $\sum_{r=0}^n v(r)$ diverges with $\sum_{r=0}^n u(r)$. For $\sum_{r=m}^{m+n} u(r) \leq k \sum_{r=m}^{m+n} v(r)$.

(ii) If $u(n+1)/u(n) \leq v(n+1)/v(n)$, then $\sum_{r=0}^n u(r)$ converges with $\sum_{r=0}^n v(r)$ and the latter diverges with the former. For $u(n) \leq \{u(0)/v(0)\}v(n)$.

Cauchy's first test. If for some constant $k, u(n) \leq k^n < 1$, then $\sum_{r=0}^n u(r)$ converges (by the foregoing test); if $u(n) \geq 1$, for $n = n(r)$, where $n(r+1) > n(r)$, then $\sum_{r=0}^n u(r)$ diverges (for $u(n)$ does not tend to zero).

D'Alembert's test. If for some $k, u(n+1)/u(n) \leq k < 1$, then $u(n+1)/u(n) \leq k^{n+1}/k^n$, and so $\sum_{r=0}^n u(r)$ converges, by (ii) above. If $u(n+1)/u(n) \geq 1$, then $\sum_{r=0}^n u(r)$ diverges, since $\sum_{r=0}^n 1$ diverges.

Kummer's test (original form). If for some constant β

$$c(n)\{u(n)/u(n+1)\} - c(n+1) \geq \beta > 0,$$

and $c(n)u(n) \rightarrow 0$, then $\sum_{r=0}^n u(r)$ converges, since

$$\beta \sum_{r=n+1}^N u(r) \leq c(n)u(n) - c(N)u(N) < c(n)u(n) \rightarrow 0.$$

Dini's form of Kummer's test, which omits the condition $c(n)u(n) \rightarrow 0$, is not valid in R (unless some additional condition is imposed, for instance that $\sum 1/c(n)$ converges, when convergence follows by the comparison test). In fact we can show that Dini's form of the test is equivalent to the assertion of the convergence of any bounded monotonic sequence. For if $s(n)$ is any strictly increasing sequence bounded by B and if $u(n) = s(n) - s(n-1)$, $v(n) = B - \sum_{r=1}^n u(r)$ and $c(n) = v(n)/u(n)$, then $c(n)\{u(n)/u(n+1)\} - c(n+1) = 1$.

Cauchy's condensation test. If $f(n)$ is steadily decreasing, then $\sum_{r=0}^n f(r)$ and $\sum_{r=0}^n 2^r f(2^r)$ converge and diverge together. For $\sum_{r=2^n}^{2^{n+1}-1} f(r) < 2^n f(2^n)$ and therefore $\sum_{r=2^n}^{2^{N+1}-1} f(r) < \sum_{r=2^n}^N 2^r f(2^r)$, whence $\sum_{r=2^n}^M f(r) < \sum_{r=2^n}^N 2^r f(2^r)$, if $2^{N+1} - 1 \geq M$.

Furthermore $\sum_{r=2^{n+1}}^{2^{n+1}} f(r) \geq 2^n f(2^{n+1})$ and so $\sum_{r=m}^{2^{N+1}} f(r) \geq \sum_{r=m}^N 2^r f(2^{r+1})$, if $2^n + 1 \geq m$.

It follows that $\sum_{r=1}^n 1/r$ is divergent, and therefore $\{1/\sum_{r=m}^{m+n} (1/r)\} \rightarrow 0$, for fixed

m . Also, $\sum_{r=1}^n 1/r^2$ converges since $\sum_{r=1}^n 1/2^r$ converges.

Raabe's tests. (i) If $u(n)/u(n+1) \geq 1 + k/n$, for some $k > 1$, then $\sum_{r=0}^n u(r)$ converges.

We observe first that $n \cdot u(n)/(n+1) \cdot u(n+1) \geq 1 + (k-1)/(n+1)$ and so, since

$$(1 + (k-1)/(n+1))(1 + (k-1)/(n+2)) \cdots \\ \cdots (1 + (k-1)/N) > (k-1) \sum_{r=n+1}^N 1/r,$$

therefore $N \cdot u(N) \leq n \cdot u(n)/(k-1) \sum_{r=n+1}^N (1/r) \rightarrow 0$, keeping n fixed. Hence

Kummer's test applies, and $\sum_{r=0}^n u(r)$ converges, since $\{n \cdot u(n)/u(n+1)\} - (n+1) \geq k-1 > 0$.

(ii) If $u(n)/u(n+1) \leq 1 + 1/n + 1/n \sum_{r=1}^n (1/r)$, then $\sum_{r=1}^n u(r)$ diverges; we prove first that $\sum_{r=1}^n \{1/r \sum_{p=1}^r (1/p)\}$ diverges. Since $\sum_{r=1}^{2^n} (1/r) < n+1$, therefore $\sum_{r=1}^n \{1/r \sum_{p=1}^{2^r} (1/p)\}$ diverges by comparison with $\sum_{r=0}^n (1/(r+1))$, and so, by Cauchy's condensation test, $\sum_{r=1}^n \{1/r (\sum_{p=1}^r (1/p))\}$ diverges. Writing $v(r) = 1/r (\sum_{p=1}^r 1/p)$, we find

$$v(n)/v(n+1) = \{1 + (n+1) \sum_{r=1}^n (1/r)\} / n \sum_{r=1}^n (1/r) \\ = 1 + 1/n + 1/n \sum_{r=1}^n (1/r) \geq u(n)/u(n+1),$$

and therefore $\sum_{r=1}^n u(r)$ diverges.

Before summarising the foregoing tests in the Gauss test we must prove the theorem:

If $\sum_{r=1}^n u(r)$ is convergent and $nu(n)$ steadily decreases to zero, then $\{\sum_{r=1}^n 1/r\} nu(n) \rightarrow 0$.

If $n \geq r$, $nu(n)/r = \{nu(n)/ru(r)\}u(r) \leq u(r)$, and so, since $\sum_{r=0}^n u(r)$

converges we can choose $n(k)$ so that $\{\sum_{r=n(k)+1}^n 1/r\}nu(n) < 1/2k$, for $n > n(k)$. Moreover since $nu(n) \rightarrow 0$, we can determine $r(k)$ such that $\{\sum_{r=1}^{n(k)} 1/r\}nu(n) < 1/2k$, for $n \geq r(k)$, whence $\{\sum_{r=1}^n 1/r\}nu(n) < 1/k$, $n \geq r(k)$, $n > n(k)$.

The Gauss test. If for some constants α, β , $u(n)/u(n+1) = \alpha + \beta/n + \theta(n)/n^2$, where $|\theta(n)| \leq M$, then by the ratio test, or Raabe's test, $\alpha > 1$ or $\alpha = 1, \beta > 1$ suffice for the convergence of $\sum_{r=0}^n u(r)$. Since $\{\sum_{r=1}^n 1/r\}(1/n) \rightarrow 0$ by the foregoing theorem, therefore we can find $N(r)$ such that $|\theta(n)|/n^2 < 1/n \sum_{r=1}^n (1/r)$, for $n \geq N(M)$, and so $\alpha < 1$, or $\alpha = 1$ and $\beta \leq 1$ suffice for the divergence of $\sum_{r=0}^n u(r)$.

This completes the preliminary survey and we come now to relative convergence.

Definitions.

The recursive function $s(m, n)$ converges in n , relative to m , if we can determine recursive functions $N(k), M(k, n, n^*)$ such that

$$|s(m, n^*) - s(m, n)| < 1/k, \text{ for all } k \geq 1, n^* \geq n \geq N(k), m \geq M(k, n, n^*).$$

$s(m, n)$ diverges in n , relative to m , if we can determine recursive functions $\lambda(n), M(n, n^*)$ and a constant $k \geq 1$ such that

$$|s(m, n + \lambda(n)) - s(m, n)| \geq 1/k, \text{ for } m \geq M(n, \lambda(n)).$$

$s(m, n) \rightarrow 0$ in n , relative to m , if we determine $N(k), M(k, n)$ such that $|s(m, n)| < 1/k$, for all $k \geq 1, n \geq N(k), m \geq M(k, n)$.

Throughout the following theorems $a(m, n), b(m, n), c(m, n)$ are positive, non-zero, recursive functions.

THEOREM 1. If for some k , $a(m, n) \leq k < 1$, for $m \geq M(n)$, then $\sum_{r=0}^n \{a(m, r)\}^r$ is convergent in n , relative to m . For $\sum_{r=n}^N a(m, r)^r \leq \sum_{r=n}^N k^r$, $m \geq \max \{M(r), n \leq r \leq N\}$.

THEOREM 1.1. If $a(m, n) \geq 1$, for $n = i(r)$, where $i(r+1) \geq r + i(r)$, and for $m \geq M(n)$ then $\sum_{r=0}^n \{a(m, r)\}^r$ diverges relative to m ; (proof trivial).

THEOREM 2. If for some $\lambda > 0$, $a(m, n) \leq \lambda b(m, n)$, for $m \geq \theta(n)$,

then the relative convergence of $\sum_{r=0}^n b(m, r)$ entails that of $\sum_{r=0}^n a(m, r)$, and the relative divergence of the latter entails that of the former. For $\sum_{r=m}^{m+n} a(m, r) \leq \lambda \sum_{r=m}^{m+n} b(m, r)$, $m \geq \max\{\theta(r), m \leq r \leq m+n\}$.

THEOREM 2.1. If there are constants h, H such that $0 < h \leq a(m, 0) \leq H$, $h \leq b(m, 0) \leq H$ and $a(m, n+1)/a(m, n) \leq b(m, n+1)/b(m, n)$, for $m \geq \theta(n)$, then the relative convergence of $\sum_{r=0}^n b(m, r)$ entails that of $\sum_{r=0}^n a(m, r)$, and the relative divergence of the latter entails that of the former. For $a(m, n) \leq (H/h)b(m, n)$, $m \geq \max\{\theta(r), 0 \leq r \leq n-1\}$.

THEOREM 3. $a(m, n+1)/a(m, n) \leq k < 1$, for $m \geq \psi(n)$, is a sufficient condition for relative convergence, since $a(m, n+1)/a(m, n) \leq k^{n+1}/k^n$, for $m \geq \psi(n)$. $a(m, n+1)/a(m, n) \geq 1$, for $m \geq \phi(n)$, suffices for relative divergence, since $\sum_{r=0}^n 1$ diverges.

THEOREM 4. If there is a constant $\beta > 0$ such that $c(m, n)\{a(m, n)/a(m, n+1)\} - c(m, n+1) \geq \beta$, for $m \geq \Gamma(n)$, and if $c(m, n)a(m, n) \rightarrow 0$ in n , relative to m , then $\sum_{r=0}^n a(m, r)$ converges relative to m .

For $\beta \sum_{r=n+1}^N a(m, r) \leq c(m, n)a(m, n)$, $m \geq \max(\Gamma(r, \beta), n \leq r \leq N-1)$.

THEOREM 5. If there is a constant $p > 1$ such that $a(m, n)/a(m, n+1) \geq 1 + p/n$, for $m \geq M(n)$ then $\sum_{r=0}^n a(m, r)$ converges relative to m .

We readily prove $n \cdot a(m, n) \rightarrow 0$ in n , relative to m , and the result then follows from the previous theorem.

THEOREM 5.1. If $a(m, n)/a(m, n+1) \leq 1 + 1/n + 1/n \sum_{r=1}^n (1/r)$, for $m \geq M(n)$, then $\sum_{r=0}^n a(m, r)$ diverges relative to m . (Proof the same as for ordinary divergence.)

THEOREM 6. If $f(m, n)$ is monotonic decreasing in n , relative to m , i. e. if there is a recursive $M(n, N)$ such that $f(m, N) < f(m, n)$, for $N > n$, $m \geq M(n, N)$, then $\sum_{r=0}^n f(m, r)$, $\sum_{r=0}^n 2^r f(m, 2^r)$ converge and diverge together, relative to m . (Proof as for ordinary convergence.)

THEOREM 7. If $a(m, n)/a(m, n+1) = \alpha + \beta/n + \theta(m, n)/n^2$,

$m \geq \Phi(n)$, $|\theta(m, n)| < M$, then $\alpha > 1$, or $\alpha = 1$, $\beta > 1$, are sufficient conditions for relative convergence, and $\alpha < 1$, or $\alpha = 1$, $\beta \leq 1$ are sufficient for relative divergence. This follows from Theorems 3, 5, 5.1.

We proceed next to sharpen Theorem 7, proving first a series of preparatory theorems.

THEOREM 8. If $a > b$, $nb^{n-1} < (a^n - b^n)/(a - b) < na^{n-1}$, $n \geq 2$.

Proof. If $x > 1$, $n > 1$, then $\sum_{r=0}^n x^r > n + 1$ (proof by induction), and therefore $(x^{n+1} - 1)/(x - 1) > n + 1$, whence, taking $x = a/b$,

$$(a^{n+1} - b^{n+1})/(a - b) > (n + 1)b^n.$$

Similarly, if $x < 1$, $n > 1$, then $\sum_{r=0}^n x^r < (n + 1)$ and therefore $(1 - x^{n+1})/(1 - x) < n + 1$, and the second half of the theorem follows by taking $x = b/a$.

THEOREM 8.1. If α is positive, and p, q are positive integers, $p > q$, then $(1 + \alpha)^p > \{1 + (p/q)\alpha\}^q$.

This is a simple consequence of a special case of the famous theorem of the means, viz.

$(ma + nb)^{m+n}/(m + n) > a^m b^n$, $a \neq b$, m, n positive, non-zero integers. (There are several algebraic proofs of the theorem of the means which are valid in the present system, for instance Cauchy's proof as it is given in [4]).

Take $m = q$, $m + n = p$, $a = 1 + (p/q)\alpha$, $b = 1$ and the theorem follows. Similarly, if $p < q$, taking $m = p$, $m + n = q$, $b = 1$, $a = 1 + \alpha$, we have $(1 + \alpha)^p < \{1 + (p/q)\alpha\}^q$.

THEOREM 9. If p, q, x are integers, $x > 1$, $q > 1$, $p \geq 1$, and if x_k is the greatest integer such that $(x_k)^q \leq x^p \cdot 2^{kq}$, then $x_k/2^k$ is convergent.

Proof. Since $(2x_k)^q \leq x^p \cdot 2^{(k+1)q}$, therefore $x_{k+1} \geq 2x_k$ and so $x_k/2^k$ is monotonic increasing, and since x_0 is the greatest integer such that $(x_0)^q \leq x^p$, and $1^q < x^p$, then $x_0 \geq 1$ and so $x_k > 2^k$. From $(x_k)^q \leq x^p \cdot 2^{kq} < x^{pq} \cdot 2^{kq}$ it follows that $x_k < x^p \cdot 2^k$. Hence

$$\begin{aligned} 0 \leq x^p - (x_k/2^k)^q &< \{(x_k + 1)^q - (x_k)^q\}/2^{kq} < q(x_k + 1)^{q-1}/2^{kq} \\ &\quad \text{(by Theorem 8)} \\ &< q\{x^p + 1/2^k\}^{q-1}/2^k \leq q\{x^p + 1\}^{q-1}/2^k \rightarrow 0, \end{aligned}$$

i. e. $(x_k/2^k)^q$ converges to x^p .

² Since $t^q - 1 = \sum_{r=0}^{q-1} t^r (t - 1)$, it follows that if $t > 0$, then $t \geq 1$ according as $t^q \geq 1$.

Writing $x_k/2^k = y_k$, so that y_k is monotonic increasing and y_k^q converges, then, since $\{(y_{k+r})^q - (y_k)^q\}/\{y_{k+r} - y_k\} > q(y_k)^{q-1} > q$, i. e. $y_{k+r} - y_k < \{(y_{k+r})^q - (y_k)^q\}/q$, so that y_k converges, which completes the proof.

Definition. We define, for $p \geq 1$, $q > 1$, $x > 1$, $\chi(x, p, q, k) = x_k/2^k$ (where x_k is given alone) and $\chi(1, q, p, k) = 1$.

THEOREM 10. $\sum_{r=1}^n 1/\chi(r, p, q, k)$ is convergent in n , relative to k , if $p > q$ and divergent in n , relative to k , if $p \leq q$.

Since $\{\chi(x, p, q, k)\}^q \rightarrow x^p$, therefore, writing $a_k = \chi(x+1, p, q, k)/\chi(x, p, q, k)$, we have if $p > q$, $(a_k)^q \rightarrow (1+1/x)^p > (1+p/qx)^q$ by Theorem 8.1; the convergence of a_k determines a recursive function $Q(x)$ such that $|(1+1/x)^p - (a_k)^q| < (1+1/x)^p - (1+p/qx)^q$ for $k \geq Q(x)$, and therefore $(a_k)^q > (1+p/qx)^q$, i. e. $a_k > 1+p/qx$ for $k \geq Q(x)$. It follows, by Theorem 5, that $\sum_{r=1}^n 1/\chi(r, p, q, k)$ converges relative to k , if $p > q$.

If $p = q$, $\chi(n, p, q, k) = n$, and so $\sum_{r=1}^n 1/\chi(r, p, q, k)$ diverges. If $p < q$, $a_k < 1+p/qx$ for $k \geq Q^*(x)$, whence, by Theorem 5.1, $\sum_{r=1}^n 1/\chi(r, p, q, k)$ diverges relative to k .

THEOREM 11. If $\phi(n, p, q, k) = n/\chi(n, p, q, k)$ then, for $p > q$, $\phi(n, p, q, k)$ is monotonic decreasing to zero, relative to k .

For

$$\begin{aligned} \phi(n, p, q, k)/\phi(n+1, p, q, k) &> \{n/(n+1)\}\{1+p/qn\} \\ &= 1 + ((p/q) - 1)/(n+1) \end{aligned}$$

if $k \geq Q(n)$, $p > q$, so that $\phi(n, p, q, k)$ is monotonic decreasing, relative to k , and further, if $N > n$,

$$\phi(N, p, q, k)/\phi(n, p, q, k) < 1/\prod_{r=n+1}^N \{1 + ((p/q) - 1)/r\}$$

$$< 1/((p/q) - 1) \sum_{r=n+1}^N (1/r) \rightarrow 0,$$

$$k \geq \max \{Q(r), n \leq r \leq N-1\},$$

i. e. $\phi(N, p, q, k) \rightarrow 0$ relative to k .

THEOREM 11.1. If $p > q$, $\{\sum_{r=1}^n (1/r)\}n/\chi(n, p, q, k) \rightarrow 0$, relative to k .

If $n > r$, by Theorem 11,

$$\phi(n, p, q, k)/\phi(r, p, q, k)\chi(r, p, q, k) < 1/\chi(r, p, q, k)$$

for $k \geq \max\{Q(m), r \leq m \leq n-1\}$. Since $\sum_{r=1}^n 1/\chi(r, p, q, k)$ converges relative to k , if $p > q$, given $x \geq 1$, we can determine $n(x)$ and $\delta(n, x)$ such that, $\sum_{r=n(x)}^n 1/\chi(r, p, q, k) < 1/2x$, for $n \geq n(x)$, $k \geq \delta(n, x)$. Furthermore, since $\phi(n, p, q, k) \rightarrow 0$, relative to k , we can determine $\nu(x)$ and $\eta(x)$ such that $\{\sum_{r=1}^{n(x)} (1/r)\}\phi(n, p, q, k) < 1/2x$, for $n \geq \nu(x)$, $k \geq \eta(n, x)$. Hence

$$\{\sum_{r=1}^n (1/r)\}\phi(n, p, q, k) < 1/x,$$

$n \geq \max\{\nu(x), n(x)\}$, $k \geq \max\{\eta(n, x), \delta(n, x)\}$, $[\max\{Q(m), n(x) \leq m \leq n-1\}]$ i. e. $\{\sum_{r=1}^n (1/r)\}\phi(n, p, q, k) \rightarrow 0$ relative to k .

We are now in a position to state the analogue, for relative convergence, of the Gauss test.

THEOREM 12. If $a(n, k)/a(n+1, k) = \alpha + \beta/n + \theta(n, k)/\chi(n, p, q, k)$, for $p > q$, $|\theta(n, k)| < M$, and $k \geq A(n)$, then $\alpha > 1$ or $\alpha = 1$, $\beta > 1$ are sufficient conditions for the convergence of $\sum_{r=0}^n a(r, k)$ relative to k , and $\alpha < 1$ or $\alpha = 1$, $\beta \leq 1$ are sufficient conditions for relative divergence.

We have to consider only the cases $\alpha = 1$, $\beta > 1$; $\alpha = 1$, $\beta \leq 1$. If $\beta > 1$, since $\theta(n, k)\phi(n, p, q, k) \rightarrow 0$ relative to k , we can find N_0 , $K(n)$ such that

$$|n \cdot \theta(n, k)/\chi(n, p, q, k)| < (1/2)(\beta - 1), \text{ for } n \geq N_0, k \geq K(n),$$

and so

$$a(n, k)/a(n+1, k) > 1 + (1/2)(\beta + 1)/n, \text{ for } n \geq N_0, k \geq \max\{K(n), A(n)\},$$

which suffices for relative convergence, by Theorem 5.

Similarly, if $\beta < 1$, $a(n, k)/a(n+1, k) < 1 + (1/2)(\beta + 1)/n$, which proves relative divergence, by Theorem 5.1.

If $\beta = 1$, since $\{\sum_{r=1}^n (1/r)\}\phi(n, p, q, k) \rightarrow 0$, relative to k , therefore we can find n_0 and $J(n)$ such that $|\theta(n, k)/\chi(n, p, q, k)| < 1/n \sum_{r=1}^n (1/r)$, for $n \geq n_0$, $k \geq J(n)$, whence $\sum_{r=0}^n a(r, k)$ is divergent, relative to k , by Theorem 5.1.

Appendix.

The index laws for $\chi(n, p, q, k)$.

1. If $r > 0$, $\chi(n, pr, qr, k) = \chi(n, p, q, k)$.

Writing $n_k = 2^k \cdot \chi(n, p, q, k)$, then $n_k^{qr} \leq n^{pr} \cdot 2^{kqr}$ and $(n_k + 1)^{qr} > n^{pr} \cdot 2^{kqr}$ so that n_k is the greatest integer such that $n_k^{qr} \leq n^{pr} \cdot 2^{kqr}$, whence the result follows.

Accordingly we may write $\chi(n, p, q, k)$ in the form $\chi(n, p/q, k)$.

2. $\chi(ab, p/q, k) - \chi(a, p/q, k) \cdot \chi(b, p/q, k) \rightarrow 0$ in k .

We have $\{\chi(a, p/q, k)\}^q \leq a^p$, $\{\chi(b, p/q, k)\}^q \leq b^p$, and therefore

$$\{2^k \chi(a, p/q, k) \cdot 2^k \chi(b, p/q, k)\}^q \leq (ab)^p 2^{2kq};$$

but $2^k \chi(ab, p/q, k) + 1$ is the least integer such that

$$\{2^k \chi(ab, p/q, k) + 1\}^q > (ab)^p \cdot 2^{kq},$$

and therefore

$$2^{2k} \chi(ab, p/q, k) + 2^k > 2^k \chi(a, p/q, k) \cdot 2^k \chi(b, p/q, k),$$

i. e.

$$\chi(ab, p/q, k) - \chi(a, p/q, k) \cdot \chi(b, p/q, k) > -1/2^k.$$

Similarly

$$(\chi(a, p/q, k) + 1/2^k)(\chi(b, p/q, k) + 1/2^k) > \chi(ab, p/q, k),$$

whence

$$\begin{aligned} -1/2^k &< \chi(ab, p/q, k) - \chi(a, p/q, k) \cdot \chi(b, p/q, k) \\ &< \{\chi(a, p/q, k) + \chi(b, p/q, k) + 1/2^k\}/2^k, \end{aligned}$$

which proves the theorem, since $\chi(a, p/q, k)$, $\chi(b, p/q, k)$ converge in k .

3. If $p \geq 1$, $q > 1$, $r \geq 1$, $s > 1$, $n \geq 1$, then

$$\chi(n, p/q, k) \cdot \chi(n, r/s, k) - \chi(n, p/q + r/s, k) \rightarrow 0 \text{ in } k.$$

(Proof the same as in 2 above.)

Definition. We define, for $a \geq 1$, $b > 1$,

$$\chi_f(a, b, p/q, k) = \chi(a, p/q, k) / \chi(b, p/q, k).$$

Since $\chi(a, p/q, k)$, $\chi(b, p/q, k)$ are both convergent in k , and $\chi(b, p/q, k) > 1$, therefore $\chi_f(a, b, p/q, k)$ is convergent in k .

4. If $c \geq 1$, $\chi_f(ac, bc, p/q, k) - \chi_f(a, b, p/q, k) \rightarrow 0$ in k .

For $\chi_f(ac, bc, p/q, k) - \chi_f(a, b, p/q, k)$

$$= \frac{\chi(ac, p/q, k)}{\chi(bc, p/q, k)} - \frac{\chi(a, p/q, k)}{\chi(b, p/q, k)} = \frac{\chi(c, p/q, k)}{\chi(c, p/q, k)} \cdot \frac{\chi(a, p/q, k)}{\chi(b, p/q, k)} - \frac{\chi(a, p/q, k)}{\chi(b, p/q, k)}$$

$\rightarrow 0$, in k , by 2. Accordingly we may write $\chi_f(a/b, p/q, k)$ for $\chi_f(a, b, p/q, k)$.

5. $\chi_f(\chi(x, p/q, k), r/s, \lambda)$ is convergent in k relative to λ and

$$\chi(x, pr/qs, k) - \chi_f(\chi(x, p/q, k), r/s, \lambda) \rightarrow 0$$

in k , relative to λ .

Proof. Write $x_k = 2^k \cdot \chi(x, p/q, k)$, so that

$$\chi_f(\chi(x, p/q, k), r/s, \lambda) = \chi(x_k, r/s, \lambda) / \chi(2^k, r/s, \lambda).$$

Keeping k fixed, $\{\chi(x_k, r/s, \lambda)\}^s \rightarrow (x_k)^r$ and $\{\chi(2^k, r/s, \lambda)\}^s \rightarrow 2^{kr}$, and, further, $(x_k/2^k)^q \rightarrow x^p$, in k , wherefore we can find $K(n)$ and $L(n, k)$ such that

$$0 \leq x^{pr} - (x_k/2^k)^{qr} < 1/n, \text{ for } k \geq K(n),$$

$$|\{x_k/2^k\}^{qr} - \{\chi(x_k, r/s, \lambda) / \chi(2^k, r/s, \lambda)\}^{qs}| < 1/n, \text{ for } \lambda \geq L(n, k),$$

and so

$$|x^{pr} - \{\chi(x_k, r/s, \lambda) / \chi(2^k, r/s, \lambda)\}^{qs}| < 2/n, \text{ for } k \geq K(n), \lambda \geq L(n, k),$$

i. e. $\{\chi_f(\chi(x, p/q, k), r/s, \lambda)\}^{qs}$ is convergent to x^{pr} in k , relative to λ ; denoting $\chi_f(\chi(x, p/q, k), r/s, \lambda)$ by $\Omega(k, \lambda)$ we see that, since $x_k > 2^k$ and $\chi(y, r/s, \lambda)$ increases with y , therefore $\Omega(k, \lambda) \geq 1$, and so for any n, N , by Theorem 8, if $p \geq 1$, $|\Omega(N, \lambda) - \Omega(n, \lambda)| \leq |\Omega(N, \lambda)^p - \Omega(n, \lambda)^p|/p$, whence it follows that $\Omega(k, \lambda)$ converges in k relative to λ . Furthermore $\{\chi(x, pr/qs, k)\}^{qs} \rightarrow x^{pr}$ and so

$$\{\chi(x, pr/qs, k)\}^{qs} - \{\Omega(k, \lambda)\}^{qs} \rightarrow 0$$

in k , relative to λ ; since $\Omega(k, \lambda) \geq 1$ and $\chi(x, pr/qs, k) \geq 1$, therefore

$$|\chi(x, pr/qs, k) - \Omega(k, \lambda)| \leq |\{\chi(x, pr/qs, k)\}^{qs} - \{\Omega(k, \lambda)\}^{qs}| / qs$$

(if $q \geq 1, s \geq 1$) and so $\chi(x, pr/qs, k) - \Omega(k, \lambda) \rightarrow 0$ in k , relative to λ .

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JACOBIAN EXTENSORS.*

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1. Introduction. The purpose of the present paper is to extend the general notions of *contravariance* and *covariance* so as to include what we shall call *Jacobian-contravariance* and *Jacobian-covariance* or briefly *j-contravariance* and *j-covariance*. As the title perhaps suggests, the essence of this innovation is the introduction of quantities which follow a transformation law which is formally equivalent to that of tensor analysis but differs in content by the presence of certain coefficients X^a_ρ which depend upon the Jacobian and are analogous to the extensor coefficients $X \sup \alpha \inf \rho$. To be more specific, the new transformation equations contain, in addition to the coefficients X^a_r ($X^a_r = \partial x^a / \partial \bar{x}^r$) and $X \sup \alpha \inf \rho$ of tensor and extensor analysis, certain coefficients X^a_ρ which are related to the weighted Jacobian \underline{X} ($\underline{X} = x/\bar{x}$)^w in the same way that the coefficients $X \sup \alpha \inf \rho$ are related to the quantities X^a_r . The symbol $X \sup \alpha \inf \rho$ denotes the partial derivative of x^a with respect to \bar{x}^ρ and the relationship just mentioned is

$$X^{aa}_{\rho r} = \left(\begin{matrix} A \\ P \end{matrix} \right) X^a_r (A-P).$$

Here $A = \alpha$, $P = \rho$ and the enclosed superscripts indicate differentiation with respect to the curve parameter t . The capital indices are introduced to forestall summation.

We shall find that Jacobian extensors include weighted and absolute tensors and scalars as special cases and that their properties are so similar to those of ordinary extensors that much of the older theory may be taken over without proof. In addition, we shall show that given an ordinary connection one may develop readily its Jacobian counterpart, which we call a *Jacobian connection*. Finally, the Jacobian connections are such that they enable us to express the higher order intrinsic derivatives of weighted tensors as contractions of certain extensors. This provides a very satisfying theory of the algebraic structure of these rather complicated derived tensors.

The material to be presented falls into two rather sharply delineated subdivisions. Part A is concerned entirely with developments which are valid in any N -dimensional space which bears a coordinate system, while

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Part B presumes a space with symmetric connection. The symmetry of the connection facilitates the work by providing for the use of geodesic coordinates.

Part A.

In the present major division, we assume that there is given an N -dimensional space, a coordinate system, and the group of one-to-one differentiable coordinate transformations.

2. Notation. The notation employed is for the most part that of certain related works, particularly numbers [3], [4], [5] of the appended bibliography. Accordingly, we describe it rather briefly. Enclosed Greek and numerical indices as well as primes indicate differentiation with respect to a curve parameter t . Repeated lower case Latin and Greek indices engender summations from 1 to N and from 0 to M , respectively, unless the contrary is indicated; while repeated capital indices do *not* generate sums. Further, indices at the last of the alphabet such as $r, s, t, u, v, w, \rho, \sigma, \tau$ indicate that the symbol of which they are a part belongs to coordinate system (\bar{x}) . Thus we write x^r for \bar{x}^r . Indices at the first of the alphabet $a, b, c, d, e, \alpha, \beta, \gamma, \delta, \epsilon$ are correlated to system (x) . Finally, the symbols (x/\bar{x}) , (\bar{x}/x) denote, respectively, the Jacobians associated with the coordinate transformations $x^a = x^a(\bar{x})$, $\bar{x}^r = \bar{x}^r(x)$, while \underline{X} and \bar{X} stand for these Jacobians raised to the w power with w a given positive integer or zero, thus $\underline{X} = (x/\bar{x})^w$.

Special convention. All terms which contain symbols bearing numerical indices out of their natural range are to be given the value zero. For example $X^{(\alpha-\rho)} = 0$ whenever ρ exceeds α , for we take the range of indices which indicate the order of differentiation to be zero and the positive integers instead of extending this range by a definition.

3. The Jacobian symbols X^a_ρ , X^ρ_a . The coefficients of the components in the transformation equation for Jacobian extensors are defined as follows:

$$\text{Definition 3.1. } X^a_\rho = \binom{A}{P} \underline{X}^{(A-P)} \quad X^\rho_a = \binom{P}{A} \bar{X}^{(P-A)}$$

$$A = \alpha, P = \rho.$$

With regard to this definition, it should be borne in mind that the new symbols vanish, by virtue of our special convention, whenever the subscript exceeds the superscript. A second and very important property of these symbols is expressed by

THEOREM 3.1. *If \bar{x} , x , and \bar{x} are any three coordinate systems of the infinite group of tensor analysis and if we correlate the indices λ , α , and ρ to these coordinate systems, then*

$$X^\lambda_\alpha X^\alpha_\rho = X^\lambda_\rho.$$

Proof. By virtue of Definition 3.1 and the fact that the effective range of α is from ρ to λ since an X vanishes whenever its subscript exceeds its superscript, we may transcribe the left member as follows

$$X^\lambda_\alpha X^\alpha_\rho = \sum_{\alpha=\rho}^{\Lambda} \binom{\Lambda}{\alpha} \binom{\alpha}{\rho} \bar{X}^{(\lambda-\alpha)} \underline{X}^{(\alpha-\rho)}.$$

We now replace the dummy index α in the right member with $\Lambda - \beta$ with β ranging from 0 to $\Lambda - \rho$. The result is

$$\sum_{\beta=0}^{\Lambda-\rho} \binom{\Lambda}{\Lambda-\beta} \binom{\Lambda-\beta}{\rho} \bar{X}^{(\beta)} \underline{X}^{(\Lambda-\rho-\beta)}$$

which upon substituting $\binom{\Lambda}{\rho} \binom{\Lambda-\rho}{\beta}$ for $\binom{\Lambda}{\Lambda-\beta} \binom{\Lambda-\beta}{\rho}$ becomes

$$\binom{\Lambda}{\rho} \sum_{\beta=0}^{\Lambda-\rho} \binom{\Lambda-\rho}{\beta} \bar{X}^{(\beta)} \underline{X}^{(\Lambda-\rho-\beta)}.$$

But by the Leibnitz rule for differentiating a product and the fact that $\bar{X}\underline{X} = (\bar{x}/x)^w (x/\bar{x})^w = (\bar{x}/\bar{x})^w = \bar{X}$, we may transform this last expression successively into the members of the equality

$$\binom{\Lambda}{\rho} (\bar{X}\underline{X})^{(\Lambda-\rho)} = \binom{\Lambda}{\rho} \bar{X}^{(\Lambda-\rho)} = X^\lambda_\rho.$$

Thus we have proved that $X^\lambda_\alpha X^\alpha_\rho = X^\lambda_\rho$ and the theorem is established.

An immediate consequence of this theorem is the

COROLLARY. $X^\rho_\alpha X^\alpha_\sigma = \delta^\rho_\sigma.$

Proof. $X^\rho_\sigma = \binom{\rho}{\sigma} 1^{(\rho-\sigma)} = \delta^\rho_\sigma.$

4. Jacobian extensors. As we have indicated in the introduction, the essential point of distinction between ordinary and Jacobian extensors is the presence of the symbols X^α_ρ in the transformation equations of the latter quantities. To illustrate, the transformation equations for the two types of Jacobian extensors of the first order of weight w ($w > 0$) and range M are as follows

$$(4.1) \quad U^\alpha = U^\rho X^\alpha_\rho;$$

$$(4.2) \quad V_\alpha = V_\rho X^\rho_\alpha.$$

The first of these quantities U will be described by saying that it is *j-contravariant*, while the quantities such as V , which follow the transformation equation (4.2), will be said to be *j-covariant*. Similarly, the single Greek indices such as the α on U^α and the α on V_α will be said to be *j-contravariant* and *j-covariant indices*, respectively. The distinction between these two equations, it will be observed, hinges on the fact that w by convention is never negative. The concept extensor when enlarged so as to include the present innovation, may be formulated as follows:

DEFINITION 4.2. Let there be given at a point P of a parameterized arc of class c^M one set of labelled numbers $E_{\beta, \delta \alpha, f}^{a, \gamma c, e}$ for each coordinate system of the infinite collection of tensor analysis. Further, let x and \bar{x} denote any two coordinate systems of this group and let $E_{\beta, \delta \alpha, f}^{a, \gamma c, e}$ and $E_{\sigma, \omega \mu, v}^{\rho, \tau t, u}$ be associated with systems x and \bar{x} , respectively. If these quantities are related according to the equation

$$(4.3) \quad E_{\beta, \delta \alpha, f}^{a, \gamma c, e} = E_{\sigma, \omega \mu, v}^{\rho, \tau t, u} X_\rho^\alpha X_\beta^\sigma X_{\tau t}^{\gamma c} X_{\delta \alpha}^{\omega \mu} X_u^e X_f^v$$

then, we shall say that they are the components of an extensor of range M and weight w which is *j-contravariant*, *j-covariant*, *excontravariant*, *excovariant*, *contravariant*, and *covariant*—each of order one.

The extension of this definition to higher orders of contravariance and covariance and the generalization to the case in which not all of the Greek indices have the same range are obvious and do not require special enunciation. It should be observed in passing, however, that if a Jacobian superscript is confined to the value zero or a Jacobian subscript to the value M (in the case of doublet indices these are tensor values), then the effect of these indices is to weight the extensor in the ordinary sense of the term; in other words, the effect is to introduce a weighted Jacobian as a multiplier. To illustrate, if α in E_f^α is confined to the value zero, then the transformation equation

$$E_f^\alpha = E_v^\rho X_\rho^\alpha X_f^v \text{ reduces to } E_f^0 = (\bar{x}/x)^{-w} E_v^0 X_f^v;$$

while the introduction of the restriction $\beta = M$ into

$$E_\beta^e = E_\sigma^u X_\beta^\sigma X_u^e \text{ yields } E_M^e = (\bar{x}/x)^w E_M^u X_u^e.$$

Thus E_f^0 is a tensor of weight $-w$, while E_M^e is a tensor of weight w , and we can see that the concept Jacobian extensor encompasses that of weighted tensor as a special case.

Reduced range. Obviously, the property just noted admits of an imme-

diate extension since it is true in general that X_ρ^α vanishes whenever ρ exceeds α . Thus, in particular, we see from the transformation equation for E_β^α , namely, $E_\beta^\alpha = E_\sigma^\rho X_\rho^\alpha X_\beta^\sigma$, that the components of E_β^α which have $\alpha \leq \theta$ and $\beta \geq \Phi$ are independent of the components in the other coordinate systems which do not meet the corresponding restrictions.

Transitive property. Since $X_\rho^\alpha X_\beta^\rho = \delta_\beta^\alpha$ and $X_\alpha^\rho X_\lambda^\alpha = X_\lambda^\rho$, it follows at once that the inverse of Equation 4.3 is of the same type as the original and that the transformation law for Jacobian extensors is transitive.

Invariance, sums, products, and contractions. From the fact that 4.3 is linear and homogeneous it follows as in tensor analysis that extensor equations of the Jacobian type are invariant equations, and that the familiar rules regarding the sums and products of tensors have their counterparts in the new theory. Similarly, as in extensor analysis, there are $M + 1$ contractions of a mixed second order Jacobian extensor. The form of these contractions is given by the following statement:

THEOREM 4.1. *If E_β^α is a Jacobian extensor of the type indicated by the indices, then for each fixed value of θ from 0 to M , inclusive, $\sum_{\alpha=\theta}^M \binom{\alpha}{\theta} E^{\alpha-\theta}_\alpha$ is an absolute scalar or invariant.*

Proof. Applying successively the transformation equation for Jacobian extensors, Definition 3.1, and the formula

$$\binom{\alpha}{\theta} \binom{\alpha-\theta}{\rho-\theta} \binom{\alpha}{\rho}^{-1} = \binom{\rho}{\theta},$$

we get the chain of equalities

$$\begin{aligned} \sum_{\alpha=0}^M \binom{\alpha}{\theta} E^{\alpha-\theta}_\alpha &= \sum_{\alpha, \rho=\theta}^M \binom{\alpha}{\theta} E^{\rho-\theta}_\sigma X^{\alpha-\theta}_{\rho-\theta} X^\sigma_\alpha \\ &= \sum_{\alpha, \rho=\theta}^M \binom{\alpha}{\theta} \binom{\alpha-\theta}{\rho-\theta} \binom{\alpha}{\rho}^{-1} E^{\rho-\theta}_\sigma X^\alpha_\rho = \sum_{\rho=\theta}^M \binom{\rho}{\theta} E^{\rho-\theta}_\sigma \delta^\sigma_\rho = \sum_{\rho=\theta}^M \binom{\rho}{\theta} E^{\rho-\theta}_\rho, \quad \alpha \geq \rho, \end{aligned}$$

and the theorem is established.

Remark. This proof is obviously sufficiently typical to enable us to infer the corresponding theorem for higher order extensors. The additional indices whether single Greek, doublet, or single Latin would call for only trivial modification.

Remark. This contraction process, except for the cases $\theta = 0$, $\theta = M$, is more complicated than that of tensor analysis. However, it will be found to be easy to reproduce if we note that the essential point is the pattern

of α and θ in the expression $\sum_{\alpha=\theta}^M \binom{\alpha}{\theta} E^{\alpha-\theta}_\alpha$. Strictly as a memory device, this pattern may be regarded as generated by rotating the $\alpha\theta$ in $E^{\alpha-\theta}$ first clockwise into the binomial coefficient $\binom{\alpha}{\theta}$ and then counterclockwise to the summation symbol, thus $\sum_{\alpha=\theta}$. With this in mind, we denote $\sum_{\alpha=\theta}^M \binom{\alpha}{\theta} E^{\alpha-\theta}_\alpha$ by the abridged symbol $RE^{\alpha-\theta}_\alpha$. In this notation the contraction theorem asserts that $RE^{\alpha-\theta}_\alpha \dots$ is an extensor of the type indicated by the remaining indices.

Quotient law. Following the pattern of the parent theories, there is associated with the contraction theorem a quotient law involving a partly arbitrary Jacobian extensor A . The degree of arbitrariness is of course that required in the proof. The theorem in question is as follows:

THEOREM 4.2. *If $E^{\alpha-\theta}_\alpha \dots$ is independent of A , and if for some admissible value of θ , $RE^{\alpha-\theta}_\alpha \dots A_\alpha$ is an extensor of the type indicated by the free indices for arbitrary choice of the Jacobian extensor A_α , then the same may be asserted of $E^{\alpha-\theta}_\alpha \dots$ for α in the reduced range 0 to $M - \theta$.*

Proof. As in the case of tensors and ordinary extensors the proof consists of merely writing out the given transformation equation, "factoring out" A_α and then assigning A_α certain special values. The details are as follows. We have given that for some fixed value of θ

$$RE^{\alpha-\theta}_\alpha \dots A_\alpha - RE^{\rho-\theta}_\rho \dots A_\rho X = 0,$$

if X represents the set of Jacobian symbols and partial derivatives that are tied into the missing indices on \bar{E} . Replacing A_ρ with $A_\alpha X^\alpha_\rho$ (the range of α here may be taken to be θ to M since $\rho \geq \theta$), we get

$$[RE^{\alpha-\theta}_\alpha \dots - RE^{\rho-\theta}_\rho \dots X^\alpha_\rho] A_\alpha = 0.$$

We now assign A_α the special values $\delta^{\phi+\theta}_\alpha$, $\Phi + \theta \leq M$, and simultaneously replace the dummy index ρ with $\rho + \theta$, with the new ρ ranging from 0 to $M - \theta$. The result is

$$\binom{\Phi + \theta}{\theta} E^{\Phi-\theta}_\theta \dots - \sum_{\rho=0}^{M-\theta} \binom{\rho + \theta}{\theta} E^{\rho-\theta}_\rho \dots X^{\phi+\theta}_{\rho+\theta} = 0.$$

An easy calculation will show that $\binom{P + \theta}{\theta} \binom{\Phi + \theta}{P + \theta} \binom{\Phi}{P}^{-1} = \binom{\Phi + \theta}{\theta}$.

Hence $\binom{P + \theta}{\theta} X^{\Phi+\theta}_{P+\theta} = \binom{\Phi + \theta}{\theta} X^{\phi+\theta}_\rho$ and the Theorem follows.

Remark. For $M - B$ any number from 0 to M , A_a will reduce to δ^{M-B}_a at $t = c$, if $B!A_a = [(t - c)^B]^{(M-a)}$. Later we shall encounter Jacobian extensors satisfying the relationship $A_a = \binom{M}{A} A_M^{(M-A)}$. Thus the foregoing proof will be valid if it is required, in addition, that $A_a = \binom{M}{A} A_M^{(M-A)}$ in all coordinate systems.

Remark. The corresponding proposition for A^a is

THEOREM 4.3. *If E^{\dots} is independent of the arbitrary extensor A and for some admissible value of θ , $RA^{a-\theta}E^{\dots}_a$ is an extensor of the type indicated by the free indices, then the same may be asserted of E^{\dots}_a for a ranging from θ to M .*

The proof of this theorem may be obtained from the proof of the corresponding theorem for ordinary extensors, making the obvious changes. See [3],¹ page 272.

Examples of Jacobian extensors. Obviously, as we have tacitly assumed in the preceding proof, Jacobian extensors may be manufactured by arbitrarily assigning values to the components in any one coordinate system and then defining the components in the other coordinate systems by means of the transformation law itself. In addition, we may, of course, take over without proof many of the theorems of ordinary extensor analysis because of the formal similarity between absolute and Jacobian extensors. Certain of these theorems will point the way to the construction of Jacobian extensors. As cases in point, we may cite the following propositions:

THEOREM 4.4. *If V^0 is a completely reduced Jacobian extensor and hence of range 0 (or, in other words, if V^0 is a scalar of weight $-W$), then the quantities $V^{0(a)}$, $0 \leq a \leq M$, constitute a Jacobian extensor of the range 0 to M .*

THEOREM 4.5. *If V_M is a completely reduced Jacobian extensor of the type indicated, i. e., a scalar of weight W , then the quantities V_a defined by $V_a = \binom{M}{A} V_M^{(M-A)}$, $A = a$, $0 \leq a \leq M$ are the components of a Jacobian extensor of range 0 to M .*

Remark. Since the proofs of these theorems are very short and essentially the same, it will perhaps be defensible to present one of them in place

¹ Numbers in brackets refer to references in the appended bibliography.

of reexamining the corresponding proofs for ordinary extensors. Accordingly, we give the argument for Theorem 4.4 in outline form.

Proof. From the transformation equation $V^\rho = V^a X^\rho_a$, we derive as the transformation equation for the completely reduced Jacobian extensor, the relationship $\bar{V}^\rho = V^a X^\rho_a = V^0 \bar{X}^\rho$. Differentiating ρ times by the Leibnitz rule, we get

$$V^{0(\rho)} = (V^0 \bar{X}^\rho)^{(\rho)} = \sum_{a=0}^P V^{0(a)} \binom{P}{a} \bar{X}^{(P-a)} = V^{0(a)} X^\rho_a.$$

These theorems provide the means for expanding the range of completely reduced Jacobian extensors (weighted scalars) to the full range $(0, M)$. A moment's consideration will show that similar theorems exist for higher order weighted tensors. To illustrate, the quantities $E_{\delta d}^{aa, \beta b, \gamma c}$ defined by $E_{\delta d}^{aa, \beta b, \gamma c} = \left\{ \begin{matrix} A, B, \Gamma \\ M, \Delta \end{matrix} \right\} T^{abc}_d (A+B+\Gamma-2M-\Delta)$ constitute the components of an absolute extensor provided T^{abc}_d is an absolute tensor.² Furthermore, the conversion of the fixed Jacobian subscript M in Theorem 4.5 into a full range Jacobian subscript by differentiation with respect to the parameter is, as we have pointed out, perfectly analogous to the conversion of a tensor subscript d into the doublet subscript δd in the illustration just cited. Similarly, the conversion of a Jacobian superscript zero (this indicates weight $-w$ in the usual nomenclature) into a full range Jacobian superscript α is formally exactly equivalent to the conversion of a tensor superscript a into the doublet superscript αa . Accordingly, we may take over without proof the following theorems from [5], pp. 335, 336.

THEOREM 4.6. *If $T^{0, bc}$ is a second order contravariant tensor of weight $-w$ and the necessary derivatives exist, then the quantities defined by*

$$E_{\delta d}^{aa, \beta b, \gamma c} = \left[\begin{matrix} A, B, \Gamma \\ M \end{matrix} \right] T^{0bc} [A+B+\Gamma-2M], \quad A = \alpha, B = \beta, \Gamma = \gamma,$$

are the components of an extensor which is j -contravariant of order 1 and excontravariant of order 2.

Remark. To make the complete formal equivalence of this theorem with Theorem 2.1 of [5] evident, we present for comparison the following

Proof. The quantity $(T^{0, bc} A_M B_b C_c)^{(M)}$ is an invariant for arbitrary choice of the scalar of weight w , A_M , and the covariant vectors B and C .

² See reference [5] pages 334-336.

Dropping the original indices, performing the indicated differentiation, and finally replacing α with $M - \alpha$, etc. we get

$$\begin{aligned} (T A B C)^{(M)} &= \sum \binom{M}{\alpha, \beta, \gamma} A^{(\alpha)} B^{(\beta)} C^{(\gamma)} T^{(M-\alpha-\beta-\gamma)} \\ &= \sum \binom{M}{M-\alpha, M-\beta, M-\gamma} \binom{M}{\alpha}^{-1} \binom{M}{\beta}^{-1} \binom{M}{\gamma}^{-1} \binom{M}{\alpha} A_{M^{(M-\alpha)}} \binom{M}{\beta} B_{M^{(M-\beta)}} \\ &\quad \binom{M}{\gamma} C_{M^{(M-\gamma)}} T^{0, bc(a+\beta+\gamma-2M)} = E^{a, \beta b, \gamma c} A_{\alpha} M_{\beta b} C_{\gamma c}, \end{aligned}$$

and the theorem follows from the quotient laws. The summation on α, β, γ is from 0 to M throughout and the quantities $A_{\alpha}, B_{\beta}, C_{\gamma}$ denote, respectively, the extensors

$$\binom{M}{A} A_{M^{(M-A)}}, \binom{M}{B} B_{M^{(M-B)}}, \binom{M}{\Gamma} C_{M^{(M-\Gamma)}}.$$

THEOREM 4.7. If T^{abc}_M is a third order contravariant tensor of weight w , then $E^{aa, \beta b, \gamma c}_{\delta}$, defined by the statements:

$$\begin{aligned} E^{aa, \beta b, \gamma c}_{\delta} &= \left\{ \begin{matrix} A, B, \Gamma \\ M, \Delta \end{matrix} \right\} T^{abc}_M (A+B+\Gamma-2M-\Delta), \\ \left\{ \begin{matrix} A, B, \Gamma \\ M, \Delta \end{matrix} \right\} &= \frac{A!B!\Gamma!}{M!M!\Delta!(A+B+\Gamma-2M-\Delta)!} \text{ or } 0 \end{aligned}$$

according as $A+B+\Gamma-2M-\Delta \geq 0$ or < 0 , is an extensor of excontravariant order three and j -covariant order one.

Remark. The multiplier of M in $(A+B+\Gamma-2M-\Delta)$ and the number of times $M!$ occurs in the denominator of the preceding fraction is one less than the number of doublet superscripts on E . To illustrate, if there are q top numbers in $\left\{ \begin{matrix} A, B, \dots \\ M, \Gamma, \Delta \end{matrix} \right\}$, then the bottom number M should be regarded as occurring $(q-1)$ times, and thus the symbol $\{ \}$ is equal to $(A!B!\dots) \div (M!^{q-1}\Gamma!\Delta![A+B+\dots-(q-1)M-\Gamma-\Delta]!)$.

THEOREM 4.8. If $T^{ab\dots}_{M, d}$ is contravariant of order q , of weight w , and covariant of order one, then $E^{aa, \beta b\dots}_{\gamma, \delta d}$ defined by

$$E^{aa, \beta b\dots}_{\gamma, \delta d} = \left\{ \begin{matrix} A, B, \dots \\ M, \Gamma, \Delta \end{matrix} \right\} T^{ab\dots}_{M, d} [A+B+\dots-(q-1)M-\Gamma-\Delta]$$

is an extensor which is extravariant of order q , j -covariant of order one, and excontravariant of order one.

THEOREM 4.9. If $T_{M, bc}$ is a second order covariant tensor of weight w

and if $E_{a,\beta b,\gamma c}$ denotes $\binom{M}{A, B, \Gamma} T_{M, bc}^{... (M-A-B-\Gamma)}$, then $E_{a,\beta b,\gamma c}$ is an extensor of j -covariant order one and excovariant order two.

Remark. These theorems are sufficiently typical to indicate the general situation. They may be proved directly or by following the method used in establishing Theorem 4.6, or by differentiating the transformation equations of the given tensors by the product rule of Leibnitz. Perhaps it should be remarked in passing that the treatment of subscripts on T is much simpler than the treatment of superscripts. In the former case it is not necessary to change the associated Greek index, and to introduce the reciprocals of the binomial coefficients. To illustrate, the proof of Theorem 4.9 may be read out of the following equation:

$$(T_{M, bc} A^a B^b C^c)^{(M)} = \sum \binom{M}{\alpha, \beta, \gamma} A^{\alpha(a)} B^{\beta(b)} C^{\gamma(c)} T_{M, bc}^{... (M-\alpha-\beta-\gamma)}.$$

Similarly, by expanding the indicated derivatives by means of the multi-product rule and proceeding as in the proofs of the foregoing theorems, we obtain the formulas:

$$(4.4) \quad (T^{0, bc} B_b C_c)^{(a)} = E^{a, \beta b, \gamma c} B_{\beta b} C_{\gamma c},$$

$$(4.5) \quad \binom{M}{\Delta} (T^{abc} A_a B_b C_c)^{(M-\Delta)} = E^{aa, \beta b, \gamma c} A_{aa} B_{\beta b} C_{\gamma c},$$

$$(4.6) \quad \binom{M}{\Gamma} (T^{ab...} A_a B_b \dots D^d)^{(M-\Gamma)} = E_{\gamma, \delta d}^{aa, \beta b, \dots} A_{aa} B_{\beta b} \dots D^{\delta(d)},$$

$$(4.7) \quad \binom{M}{A} (T_{M, bc} B^b C^c)^{(M-A)} = E_{a, \beta b, \gamma c} B^{\beta(b)} C^{\gamma(c)}.$$

Finally, we note for future reference that if the extensors A_{aa} , $B^{\beta(b)}$ etc appearing in the right members of these relationships are all expressible by extensor equations of the type $V^{a(a)} = V^b L^{aa}_b$ or $V_{aa} = V_b L^b_{aa}$, then we have at once:

$$(4.8) \quad (T^{0, bc} B_b C_c)^{(a)} = E^{a, \beta b, \gamma c} L^a_{\beta b} L^g_{\gamma c} B_f C_g;$$

$$(4.9) \quad \binom{M}{\Delta} (T^{abc} A_a B_b C_c)^{(M-\Delta)} = E^{aa, \beta b, \gamma c} L^e_{aa} L^f_{\beta b} L^h_{\gamma c} A_e B_f C_h;$$

$$(4.10) \quad \binom{M}{\Gamma} (T^{ab...} A_a B_b \dots D^d)^{(M-\Gamma)} = E_{\gamma, \delta d}^{aa, \beta b, \dots} L^e_{aa} L^f_{\beta b} \dots L^h_{\delta d} A_e B_f \dots D^h;$$

$$(4.11) \quad \binom{M}{A} (T_{M, bc} B^b C^c)^{(M-A)} = E_{a, \beta b, \gamma c} L^{\beta b}_a L^{\gamma c}_b L^d C^e.$$

In our application of these formulas the quantities L will be taken to be the extended components of connection introduced in the major division immediately following.

Part B.

The foregoing developments have been presented against the background of a space devoid of the structure imposed by an affine connection, and are of course valid regardless of the nature of any connection that might be added subsequently. In the two sections which follow, numbers five and six, we shall apply the preceding theory to the fundamental problem of constructing tensors from weighted tensors by differentiation with respect to the curve parameter—or, in a word, the problem of intrinsic differentiation. Accordingly, we now assume that we have given in addition to our space a set of components of connection of class C^M and, in order to secure the possibility of introducing geodesic coordinates, we suppose further that these components of connection are symmetric three index symbols L^a_{bc} , $L^a_{bc} = L^a_{cb}$. From the quantities L^a_{bc} , we can of course derive the two index components of connection L^a_b , $L^a_b = L^a_{bc}x'^c$ and finally the extended components of connection L^{aa}_b , L^b_{aa} . These latter quantities may be constructed from the following formulas: $L^c_{aa} = \binom{M}{A} I^c_{aa}$, $A = \alpha$; $I^c_{a-1'a} = I'^c_{aa} + I^b_{aa}L^c_b$, $I^c_{Ma} = \delta^c_a$; $L^{a+1,a}_c = L^{aa'}_c - L^{aa}_bL^b_c$, $L^{aa}_c = \delta^a_c$. The extended components of connection are extensors and are involved in iterated intrinsic differentiation. In addition, they enable us to express the derivatives of the components of a vector which is equipollent along a given curve in terms of the components of the vector. Specifically, if V^a and V_a are the components of vectors defined and equipollent along a curve C , then ³ $V^{a(a)} = V^bL^{aa}_b$ and $V_{aa} = V_bL^b_{aa}$, $V_{aa} = \binom{M}{A} V_{a^{(M-A)}}$, $A = \alpha$, A not summed. Thus we note in passing that formulas (4.8)-(4.11) inclusive are valid if the vectors A, B, C are each equipollent along the curve in question. Perhaps it should be emphasized that equipollence throughout space is not required.

5. Jacobian connections. In addition to the foregoing methods for constructing Jacobian extensors from weighted scalars and tensors, it develops that it is possible to construct the new quantities from certain absolute

³ These equations were obtained by B. B. Townsend and H. V. Craig in a paper that has not yet been submitted for publication. They may be proved readily by induction.

extensors by merely contracting components through their Latin indices and then multiplying by the appropriate invariants, $1/N$ and w . To convince ourselves that such procedures may be fruitful, it will suffice to compare the transformation equations for Jacobian and absolute extensors with M equal to unity. These equations expanded on the Greek superscripts are:

$$J^\rho_M = J^0_M \underline{X} X^\rho_0 + J^1_M \underline{X} X^\rho_1; \quad E^{\rho r}_s = E^{0a}_b X^{\rho r}_{0a} X^b_s + E^{1a}_b X^{\rho r}_{1a} X^b_s.$$

In either equation if $\rho = 0$, the second term drops out and the Jacobian equation becomes the transformation equation for an absolute invariant while the second equation reduces to the transformation equation of an absolute tensor of the right type to yield an invariant on contraction. When $\rho = 1$, the first equation becomes

$$(5.1) \quad \bar{J}^1_M = J^0_M \underline{X} X' + J^1_M = J^0_M w \ln'(\bar{x}/x) + J^1_M,$$

while the second may be written

$$(5.2) \quad E^{1r}_s = E^{0a}_b X^{r'}_a X^b_s + E^{1a}_b X^b_s X^r_a.$$

Now, according to a well known formula, $\ln'(\bar{x}/x) = X^{r'}_a X^a_r$ (X^a_r is the normalized cofactor of X^r_a). Consequently, if E^{0a}_b is the Kronecker delta δ^a_b and we multiply equation (5.2) by w and contract, the result $wE^{1r}_r = w \ln'(\bar{x}/x) + wE^{1a}_a$ may be identified with (5.1), if we adopt the definitions: $\bar{J}^1_M = wE^{1r}_r$, $J^1_M = wE^{1a}_a$, $J^0_M = 1$ ($= 1/N \delta^a_a = 1/NE^{0a}_a$). To recapitulate, we assert

THEOREM 5.1. *If E^{aa}_b is an absolute extensor of the type indicated by its indices for $M = 1$ and if further $E^{0a}_b = \delta^a_b$, then the quantities J^a_M defined by $J^0_M = 1/NE^{0a}_a$, $J^1_M = wE^{1a}_a$, constitute the components of a Jacobian extensor.*

The corresponding proposition for extensors of the type E^{b}_{aa} is as follows:

THEOREM 5.2. *If E^{b}_{aa} is an extensor with $M = 1$, $E^{b}_{1a} = \delta^b_a$ then the quantities J^0_a defined by*

$$J^0_a = I(A)E^a_{Aa}, \quad A = \alpha, \quad I(0) = W, \quad I(1) = 1/N,$$

are the components of a Jacobian extensor.

The proof of this theorem is similar to that of number (5.1), accordingly we omit it.

Remark. The connection extensors L^{aa}_b and L^b_{aa} , with $M = 1$, are examples of extensors of the types involved in the two preceding theorems.

These extensors it will be recalled are defined by the equations $L^{a_b} = L^{a_{1b}} = \delta^a_b$, $L^{1a_b} = -L^a_b = -L^a_{bc}x'^c$, $L^a_{0b} = L^a_b$. Here the L^a_{bc} are the Christoffel symbols or, more generally, the components of a symmetric connection. Symmetry is desirable since it allows the introduction of geodesic coordinates.

DEFINITION 5.1. *The Jacobian extensors J^a_M and J^0_a which are derived from the ordinary connection extensors in accordance with the formulas:*

$$J^a_M = I(A)L^a_a, \quad J^0_a = I(A)L^a_{Aa}, \quad A = \alpha, \quad M = 1,$$

with $I(A) = 1/N$ or w according as the index A on the operand L is a tensor index or a nontensor index, will be called the Jacobian connection extensors. The table of values for these symbols is $J^0_1 = 1$, $-J^1_1 = J^0_0 = wL^a_a$.

Remark. It will be found that the Jacobian connection extensors play a rôle in the intrinsic differentiation of weighted tensors which is entirely analogous to that played by the ordinary connection extensors in the differentiation of absolute tensors.

As an illustration of this remark, let us consider the following proposition:

THEOREM 5.1. *If S_M and S are scalars of weights w and $-w$, respectively, and if $S_a = \binom{M}{A} S_M^{(M-A)}$, $S^a = S^{(a)}$, then the contractions $J^a_M S_a$ and $J^0_a S^a$, $M = 1$, yield the intrinsic derivatives of S_M and S .*

Proof. $J^a_M S_a = J^0_M S_0 + J^1_M S_1 = S'_M - S_M w L^a_a = \delta S_M / \delta t$, and $J^0_a S^a = S' + S w L^a_a = \delta S / \delta t$.

Extensive differentiation and the extended Jacobian connection. A moment's consideration will show that the processes *upper* and *lower extensive differentiation*, which were defined in [4], have their counterparts in the present theory. Thus, if E^0_δ is a Jacobian extensor of the type indicated by its indices, S^0 is a scalar of weight $-w$, while S^δ denotes $S^{0(\delta)}$, then $E^0_\delta S^\delta$ and its intrinsic derivative $I(E^0_\delta S^\delta)$ are scalars of weight $-w$ —(the operator I indicates intrinsic differentiation). Consequently, by the quotient law the multipliers of S^δ in $I(E^0_\delta S^\delta)$ constitute the components of a Jacobian extensor of range $M + 1$. We denote this derived extensor by the symbol $DE|^\circ_\delta$ and call it the *upper extensive derivative* of E^0_δ . The details of this procedure are as follows:

$$\begin{aligned} I(E^0_\delta S^\delta) &= E^0_\delta S^{\delta+1} + E^{0'}_\delta S^\delta + E^0_\delta w L S^\delta \\ &= \sum_{\delta=0}^M (E^0_{\delta-1} + E^{0'}_\delta + E^0_\delta w L) S^\delta + E^0_M S^{M+1}, \quad L = L^a_a. \end{aligned}$$

Thus we are led to formulate

DEFINITION 5.2. If E^0_δ is a Jacobian extensor, then the quantities $DE|^\circ_\delta$ given by the equations

$$DE|^\circ_0 = E^{\circ\prime}_0 + E^0_0 wL; \quad DE|^\circ_\delta = E^0_{\delta-1} + E^{\circ\prime}_\delta + E^0_\delta wL, \quad DE|^\circ_{M+1} = E^0_M,$$

will be called the upper extensive derivative of E^0_δ .

Remark. The process upper extensive differentiation extends the range of the variable index. Except for the additional term, it is similar to intrinsic differentiation of a contravariant vector.

The lower extensive derivative may be obtained by a similar procedure. The starting point is the intrinsic derivative of the weighted scalar $E^a_M S_a$. Here $S_a = \binom{M}{A} S_M^{(M-A)}$ with S_M a scalar of weight w . The intrinsic derivative of $E^a_M S_a$ will involve linearly the derivatives of S_M from order zero to order $M+1$. Application of the quotient law will then yield a new extensor which involves the total derivative of E and the connection quantity L . In order to carry out this procedure it will be necessary first to evaluate S_a and S'_a in terms of S^*_a , the corresponding quantities for range $M+1$, with $S^*_{M+1} = S_M$. These formulas

$$(5.3) \quad (M+1)S'_a = (M-A+1)S^*_a, \quad (M+1)S_a = (A+1)S^*_{A+1}, \\ A = \alpha,$$

may be verified readily. Applying them to the intrinsic derivative, which is given by the equality,

$$I(E^a_M S_a) = E^a_M S'_a + E^{a\prime}_M S_a - E^a_M S_a wL,$$

we get the relationship

$$(M+1)I(E^a_M S_a) = \sum_{a=0}^M (M-\alpha+1)E^a_M S^*_a \\ + \sum_{a=1}^{M+1} \alpha(E^{a-1\prime}_M - E^{a-1}_M wL)S^*_a \\ = \sum_{a=0}^{M+1} [(M-\alpha+1)E^a_M + \alpha(E^{a-1\prime}_M - E^{a-1}_M wL)]S^*_a.$$

The conclusion is that the quantities in the bracket constitute the components of an extensor of range zero to $M+1$.

DEFINITION 5.3. If E^a_M is a Jacobian extensor of the type indicated by the free indices then the quantities $D_1 E|^\alpha_{M+1}$ defined by $D_1 E|^\circ_{M+1} = E^0_M$; $(M+1)D_1 E|^\alpha_{M+1} = (M-\alpha+1)E^a_M + \alpha(E^{a-1\prime}_M - E^{a-1}_M wL)$ will be called the lower extensive derivative of the original extensor E^a_M .

Remark. Again the range of the derived extensor is *one greater* than that of the original extensor.

Since any absolute invariant is a Jacobian extensor of the type E^0_M , it follows that we may apply *upper* and *lower extensive differentiation* to any such quantity. As an illustration, we complete the extensive derivatives of unity. Thus, if we take the $M=0$ and $E^0_M=1$ and apply Definition 5.2, we get $DE|_1^0=1$; $DE|_0^0=wL$, the components of the Jacobian connection extensor J^0_a . Similarly if we apply *lower extensive differentiation* (Definition 5.3), we obtain the results $D_1E|_1^0=1$; $D_1E|_1^1=-wL$ or $D_1E|_1^1=J^a_1$. As a recapitulation, we state

THEOREM 5.2. *The upper and lower extensive derivatives of unity are equal respectively to J^0_a and J^a_1 , the components of the Jacobian connection extensors.*

From the development of extensive differentiation, it is obvious that if $E^0_a S^a$ gives the M -th intrinsic derivative of S^0 , then $DE|_a^0 S^a$, with a summed from zero to $M+1$, will give the intrinsic derivative of order $M+1$ of S^0 . Likewise, $D_1E|^{a_{M+1}}_a S^a$ with a summed from 0 to $M+1$ is the intrinsic derivative of $S^{*_{M+1}}$ ($S^{*_{M+1}}=S_M$) of order $M+1$, if $E^a_M S_a$ is the M -th order intrinsic derivative of S_M . Consequently, since the first order extensive derivatives of unity constitute the components of the Jacobian connection extensor, we may generate quantities E , having the properties just postulated, by repeated extensive differentiation. With this as a background, we introduce

DEFINITION 5.4. *The Jacobian extensor generated by applying repeatedly upper and lower extensive differentiation to unity will be denoted by the symbols J^0_a and J^a_M , respectively, and will be referred to as the extended Jacobian connections;*

and assert

THEOREM 5.3. *If S is a scalar of weight $-w$ and S^a denotes $S^{(a)}$, then the M -th order intrinsic derivative of S is given by the relationship $I^M S = J^0_a S^a$;*

and

THEOREM 5.4.* *If S_M is a scalar of weight w and S_a denotes*

*Mr. J. M. Hurt has treated in his Master's thesis the problem of expressing certain intrinsic derivatives of weighted tensors as contractions of the appropriate extensors with the components of the absolute extended connection. In particular he investigated the case of the first order derivatives of higher order weighted tensors and the first and second derivatives of weighted vectors.

$\binom{M}{A} S_M^{(M-A)}$, then the contraction $J^a_M S_a$ yields the M -th order intrinsic derivative of S_M , IMS_M .

Because of the formal similarity of Jacobian extensive differentiation and absolute extensive differentiation, we may take over the rules given in [3] for the computation of extended connections. Specifically, to compute J^0_a , we proceed as follows: We start with $J^0_M = 1$; then we differentiate 1 and add in the product of 1 with wL ; next we take the result just obtained and differentiate it and add in its product with wL . This process is continued until we have constructed $M + 1$ terms. The final step is to multiply in the binomial coefficients from row number $M + 1$ of Pascal's triangle.

The procedure to be followed in the construction of J^a_M is somewhat simpler. We start with $J^0_M = 1$ as before, but instead of adding the products with wL we subtract these products and do not introduce the binomial coefficients. Thus to obtain any component of J^a_M after the given one, we differentiate the preceding component and subtract the product of this preceding component with wL .

As an illustration of these rules, we compute J^0_a and J^a_3 for the case $M = 3$. The calculation is as follows:

$$\begin{aligned} J^0_3 &= 1, J^0_2 = 3(1' + 1wL) = 3wL, J^0_1 = 3(wL' + w^2L^2), \\ J^0_0 &= (wL' + w^2L^2)' + (wL' + w^2L^2)wL; \\ J^0_3 &= 1, J^1_3 = 1' - wL = -wL, J^2_3 = -wL' + w^2L^2, \\ J^3_3 &= (-wL' + w^2L^2)' - (-wL' + w^2L^2)wL. \end{aligned}$$

6. The structure of the intrinsic derivatives of weighted tensors. In a preceding paper [5], the M -th order intrinsic derivatives of certain typical higher order tensors are expressed as contractions of the extended components of connection with certain derived extensors. Thus it appears that these rather complicated derivatives may be "factored" into quantities which themselves have desirable invariantive properties. We shall now show that the M -th order intrinsic derivatives of higher order *weighted* tensors admit of a similar "factorization."

The method of attack is to contract the weighted tensors with arbitrary equipollent vector fields so as to yield a weighted scalar, which we denote by S_M or S^0 according as the weight is $+w$ or $-w$, $w > 0$. We then form the M -th order intrinsic derivative of S first employing Theorem 5.4 or Theorem 5.3 and then by employing the analogue for intrinsic derivatives

of the formula for the derivative of a multiproduct. This formula may be verified readily by introducing geodesic coordinates and considering the key case $M = 1$. The extension to higher values of M follows the pattern of the parent theorem.

The specific theorems to be studied are:

THEOREM 6.1. *If $T^{0.ab}$ is a tensor of weight $-w$ and of contravariant order two and if $E^{a.\beta b.\gamma c}$ is the associated extensor (given in Theorem 4.6), then $J^0_a E^{a.\beta b.\gamma c} L^f_{\beta b} L^g_{\gamma c}$ is the M -th order intrinsic derivative of $T^{0.ab}$.*

Proof. Let S^0 denote $T^{0.bc} B_b C_c$, with B and C arbitrary, equipollent, absolute vectors. By way of Theorem 5.3 and Equation 4.8, we may write

$$I^M S^0 = J^0_a S^{0.(a)} = J^0_a E^{a.\beta b.\gamma c} L^f_{\beta b} L^g_{\gamma c} B_f C_g.$$

Similarly, from the rule for expressing derivatives of a multiproduct in terms of the derivatives of the factors, and the fact the intrinsic derivatives of B and C of order greater than zero vanish, we have

$$I^M S^0 = \sum \binom{M}{\beta, \gamma} (I^\beta B_b) (I^\gamma C_c) I^{M-\beta-\gamma} T^{0.bc} = B_f C_g I^M T^{0.fg}.$$

Consequently,

$$B_f C_g I^M T^{0.fg} = B_f C_g J^0_a E^{a.\beta b.\gamma c} L^f_{\beta b} L^g_{\gamma c},$$

and the theorem follows by way of the arbitrariness of B and C .

THEOREM 6.2. *If T^{efg}_M is a tensor of weight w and of contravariant order three, and if $E^{aa.\beta b.\gamma c}_\delta$ is the corresponding derived extensor, then $J^\delta_M E^{aa.\beta b.\gamma c}_\delta L^e_{aa} L^f_{\beta b} L^g_{\gamma c}$ is the M -th order intrinsic derivative of T^{efg}_M .*

Proof. Following the pattern of the previous proof, we employ Theorem 5.4 and Equation 4.9 and conclude that

$$I^M S_M = J^\delta_M S_\delta = J^\delta_M \binom{M}{\delta} S_{M^{(M-\delta)}} = J^\delta_M E^{aa.\beta b.\gamma c}_\delta L^e_{aa} L^f_{\beta b} L^g_{\gamma c} A_e B_f C_g.$$

Next we apply the multiproduct rule to S_M and simplify the result by taking advantage of the equipollence of the vectors A, B, C . Thus we have

$$I^M S_M = \sum \binom{M}{\alpha, \beta, \gamma} (I^\alpha A_a) (I^\beta B_b) (I^\gamma C_c) I^{M-\alpha-\beta-\gamma} T^{abc}_M = A_a B_b C_c I^M T^{abc}_M.$$

The theorem then follows by comparison of results.

THEOREM 6.3. *If $T^{ab\dots m}_d$ is contravariant of order q , of weight w , and covariant of order one, and $E^{aa.\beta b\dots \gamma \delta d}$ is the extensor presented in Theorem*

4. 8, then the contraction $J^\gamma_M E^{aa.\beta b \dots \gamma.\delta d} L^e_{aa} L^f_{\beta b} \dots L^{\delta d}_h$ is the M -th order intrinsic derivative of $T^{ef \dots}_{M, \lambda}$.

Proof. Since the salient features of the proof are the same as in the two preceding theorems, we present the argument in skeleton form only. The foundations are supplied by Theorem 5.4 and Formula 4.10. Briefly, we have, on the one hand,

$$I^M S_M = J^\gamma_M S_\gamma = J^\gamma_M \binom{M}{\gamma} S_{M \cdot (M-\gamma)} = J^\gamma_M E^{aa.\beta b \dots \gamma.\delta d} L^e_{aa} L^f_{\beta b} L^{\delta d}_h A_e B_f D^h;$$

while on the other

$$\begin{aligned} I^M S_M &= \sum \binom{M}{\alpha \beta \delta} (I^\alpha A_a) (I^\beta B_b) \dots (I^\delta D^d) I^{M-\alpha-\beta \dots \delta} T^{ab \dots}_{M, d} \\ &= A_e B_f \dots D^h I^M T^{ef \dots}_{M, \lambda}. \end{aligned}$$

THEOREM 6.4. If T_{Mbc} is a second order covariant tensor of weight w and $E_{a.\beta b.\gamma c}$ is the extensor of Theorem 4.9, then

$$J^a_M E_{a.\beta b.\gamma c} L^{\beta b}_d L^{\gamma c}_e = I^M T_{M, de}.$$

Proof. From Formula 4.11 and procedures similar to those employed in establishing the foregoing theorems, we get the equalities:

$$I^M S_M = J^a_M S_a = J^a_M \binom{M}{\alpha} S_{M \cdot (M-\alpha)} = J^a_M E_{a.\beta b.\gamma c} L^{\beta b}_d L^{\gamma c}_e B^d C^e,$$

and

$$I^M S_M = \sum \binom{M}{\beta \gamma} (I^\beta B^b) (I^\gamma C^c) I^{(M-\beta-\gamma)} T_{M, bc} = B^d C^e I^M T_{M, de}$$

and the theorem is established.

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ALGEBRAIC SYSTEMS OF POSITIVE CYCLES IN AN ALGEBRAIC VARIETY.*

By WEI-LIANG CHOW.

1. Introduction. In a paper¹ some years ago van der Waerden and the author introduced the idea of the associated form of a positive cycle. For a positive cycle of dimension d and degree n in the projective space S_m of m dimensions the associated form is a form of $d + 1$ sets of $m + 1$ indeterminates each, which has the degree n in each set of the indeterminates. If we arrange all the monomials of this type in an arbitrary but fixed order $\omega_0, \omega_1, \dots, \omega_t$, then each form of this type can be expressed in a unique way as a linear combination $\sum_{i=0}^t c_i \omega_i$ of the monomials. By assigning to each such

form the point in the projective space S_t with the (homogeneous) coordinates (c_0, c_1, \dots, c_t) , we obtain a one-to-one correspondence between all the forms of this type and all the points of S_t . Applying this in particular to the associated form, we can then assign to each positive cycle of dimension d and degree n in S_m a point in S_t ; we shall call the coordinates of this point, which are the coefficients of the associated form, the natural coordinates of the positive cycle. In this way the totality of all the positive cycles of dimension d and degree n in S_m (or in any variety contained in S_m) is represented by a set of points in S_t . The main theorem about this representation of positive cycles in S_m by points in S_t , proved in the above mentioned paper,² asserts that this set of points constitutes a bunch of varieties in S_t . An important consequence of this main theorem is that it enables us to give a precise definition of an (irreducible) algebraic system of positive cycles in S_m : If V is a variety contained in the above mentioned bunch of varieties in S_t , then the set of all the positive cycles in S_m which correspond to the points in V is called an algebraic system of positive cycles. The variety V is then called the associated variety of the algebraic system. The associated variety gives a representation of the algebraic system of positive cycles in the sense that there is a one-to-one correspondence between the

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¹ Chow and van der Waerden [5]. Following Weil [8], we shall use the expression "positive cycle" instead of "reducible variety."

² Chow and van der Waerden [5], p. 698 and p. 700.

points of V and the positive cycles of the algebraic system, and that this correspondence is of an algebraic nature. This latter statement means that there is an algebraic correspondence T between the variety V and the carrier variety U of the algebraic system³ such that to each point of V there correspond under T exactly the points of the corresponding positive cycle. This algebraic correspondence T is called the associated correspondence of the algebraic system of positive cycles. The significance of such a representation of an algebraic system of positive cycles as an algebraic variety is easily seen, for it enables us to apply the concepts and results in the theory of algebraic varieties to the study of the algebraic systems. Thus, for example, the set of all linear subspaces (i. e. positive cycles of degree 1) of a given dimension in a projective space is represented by means of their natural coordinates, which in this special case are the well-known Pluecker coordinates, as a certain variety called the Grassmann variety, and various algebraic systems of such linear subspaces are then represented as subvarieties of the Grassmann variety. It is well-known that the study of these algebraic systems of linear subspaces is greatly facilitated by their representation as such subvarieties.

However, except in very special cases such as the Grassmann variety and its subvarieties just mentioned, the usefulness of the associated variety of an algebraic system has been hitherto rather limited. The main reason for this lies in the fact that we know very little about the properties of the associated variety besides its algebraic nature and the fact that it represents the system of positive cycles in a one-to-one manner. In fact, if we consider an associated variety in general without any restriction, it cannot have by itself any special properties; for it is easily seen that any variety whatsoever can be the associated variety of a suitably chosen algebraic system of positive cycles of any given dimension. Therefore we must look for whatever special properties that might exist in the relation between the carrier variety U and the associated variety V under the associated correspondence T ; in other words, we must study the special properties of the associated correspondence T and the manner in which it connects the properties of U with those of V . In particular, it is important for the purpose of application to know under what conditions about the variety U and the correspondence T can we be sure that the associated variety V is non-singular or at least simple at a given point. For, most of the deeper concepts and results in the geometry of an algebraic variety can be applied without serious restrictions only to a

³ The carrier variety of an algebraic system is defined as the smallest variety which contains all the cycles of the system.

non-singular variety or only to a simple point or subvariety of a variety. For example, the entire intersection theory has been up to now developed only for the case when the ambient variety is simple at the intersection point or variety, and there are strong indications that such an intersection theory cannot be extended to a substantially more general class of varieties without a radical change of its character. Thus the usefulness of the Grassmann variety in the study of the geometry of systems of linear subspaces is due by no mean measure to its being non-singular. A particular case in point is the problem of the Jacobian variety of an algebraic curve, a problem which we have studied in some detail in a recent paper.⁴ We have shown there that each class of equivalent positive divisors of degree $n > 2g - 2$ on a non-singular curve C (where g is the genus of C) can be represented as a subvariety of dimension $n - g$ on the n -fold symmetric product C^n of the curve C , and that the set of all such subvarieties forms an algebraic system. The associated variety of this algebraic system would then be a Jacobian variety of the curve C , if we could prove that it is non-singular; and this requirement of being non-singular is essential, for without this property a Jacobian variety would not be of much use in the study of the geometry of the divisor classes. In the quoted paper we have solved the problem by proving that any derived normal model of the associated variety is a Jacobian variety; however, as we have indicated there, the entire results would be much more satisfactory if we could prove that the associated variety itself is already a Jacobian variety. That this is so will follow from the results of the present paper; in fact, the attempt to prove this particular result about the Jacobian variety is the original problem from which we started out and which leads us finally to the general problem considered here.

Before we go over to the exact formulation of the problem and results, we shall give a short summary of the main definitions so that there will be no ambiguity about the terminology we are going to use in this paper. This is the more necessary in view of the fact that there has been so much divergence in terminology in the recent works in the field of algebraic geometry. Although this is a rather unfortunate state of affairs, yet it is in a sense not entirely avoidable; for the different terminologies of different authors are actually more suitable for their particular purposes, and it is difficult to adopt a uniform terminology without its being too cumbersome to be convenient. In fact, we ourselves are forced to adopt in the present paper a somewhat different terminology from that of our recent paper on the Jacobian variety; for, in our present problem the consideration of "relative" varieties

⁴ Chow [4].

over a given ground field is natural and in a sense essential, while in the other paper we have only to deal with the absolute varieties.

We consider a field \mathfrak{R} called the "universal domain" which is an algebraically closed field of infinite degree of transcendency over the prime field contained in it. The set of all subfields K of the universal domain \mathfrak{R} such that \mathfrak{R} has an infinite degree of transcendency over K , has evidently the property that the intersection of any number of fields of the set is also a field of the set, and that any subfield of \mathfrak{R} which is a finite extension of any field of the set is also a field of the set. In this paper, whenever we speak of "a field," we shall always mean a field of this set; on the few occasions when we have to consider a field not in this set, we shall expressly say so and refer to such a field as an "abstract field." Any ordered set of $m+1$ elements $(x) = (x_0, x_1, \dots, x_m)$ in the universal domain \mathfrak{R} , not all zero, is called a point of the projective space S_m of dimension m , and two points are regarded as identical if their coordinates are proportional. Given any point (x) in S_m and a field K , the field $K((x))$ is the extension of K generated by the adjunction of the ratios of the $m+1$ coordinates x_0, x_1, \dots, x_m of the point (x) . We can regard this field $K((x))$ as the field generated by the adjunction of the point (x) ; it serves the same purpose as the ordinary field extension by adjunction of coordinates in the case of a point in the affine space. In similar way we can define the extension $K((x), (y), \dots)$ generated by the adjunction of several points $(x), (y), \dots$ in the same or different projective spaces. The degree of transcendency of the field $K((x))$ over K is called the dimension of the point (x) over K ; the point (x) is called algebraic over K if its dimension over K is zero, and it is called rational over K if $K((x)) = K$. Beside the elements of the universal domain \mathfrak{R} , we shall consider also elements which are not in \mathfrak{R} , mainly as symbols by means of which we can express polynomials and functions. Such an element is called an indeterminate, and an ordered set of $m+1$ indeterminates $(X) = (X_0, X_1, \dots, X_m)$ which are independent variables over \mathfrak{R} is called a set of indeterminates in S_m . We shall always denote the indeterminates by such capital Latin letters as X, Y, Z .

Let K be a field and let (X) be a set of indeterminates in S_m . Any homogeneous prime ideal \mathfrak{S} in the ring $K[X] = K[X_0, X_1, \dots, X_m]$ is said to define a variety U/K in S_m over the field K . A point (x) is said to be in the variety U/K if all the polynomials of the ideal \mathfrak{S} vanish for $(X) = (x)$. A variety W/K' over any other field K' is said to be contained in the variety U/K if every point of W/K' is in U/K ; if we have furthermore $K = K'$, then the variety W/K is said to be a subvariety of U/K . Two varieties are

identical if they are subvarieties of each other. Any point (x) in S_m determines over any field K a homogeneous prime ideal consisting of all the polynomials of $K[X]$ which vanish for $(X) = (x)$, and hence it determines a variety over K . A point (x') is called a specialization of the point (x) over K if it is contained in the variety determined by the point (x) over K ; and if the point (x) is at the same time also a specialization of (x') over K , then the two points are called generic specializations of each other over K . Any variety U/K contains a point (x) such that the variety determined by (x) over K is identical with U/K ; such a point (x) is called a generic point of U/K . Any two generic points (x) and (x') of a variety U/K are equivalent over K in the sense that the fields $K((x))$ and $K((x'))$ are K -isomorphic under the correspondence $(x) \leftrightarrow (x')$, i. e. the ratios of the corresponding coordinates are to correspond to each other. In particular, all generic points of a variety U/K have the same dimension over K ; this dimension is called the dimension of the variety U/K . A finite set of varieties is called a bunch of varieties; and any point or variety contained in any one of the varieties of the bunch is said to be contained in the bunch.

Let \tilde{K} be a field containing K . If U/K is a variety such that its defining ideal \mathfrak{S} in $K[X]$ generates also a prime ideal $\tilde{\mathfrak{S}}$ in $\tilde{K}[X]$, then the variety defined by $\tilde{\mathfrak{S}}$ over \tilde{K} is called the extension U/\tilde{K} of U/K over \tilde{K} . The extension of a variety is in a sense essentially the same as the original variety, for they not only are contained in each other but also can be defined by the same set of equations.⁵ This is the reason why we denote the extension by the same symbol U as the original variety. If a subvariety W/K of U/K has also an extension W/\tilde{K} over \tilde{K} , then W/\tilde{K} is also a subvariety of U/\tilde{K} . In order that a variety U/K has an extension U/\tilde{K} over \tilde{K} , it is necessary and sufficient that the fields \tilde{K} and $K((x))$ are linearly disjoint⁶ over K , where (x) is a generic point of U/K , free with respect to \tilde{K} over K .

Two varieties in S_m are called equivalent if they are both extensions of a third variety. It is easily seen that equivalent varieties are contained in each other and can be defined by the same set of equations; and that two varieties over the same field can only be equivalent if they are identical. We shall consider equivalent varieties as essentially the same and shall denote all of them by the same symbol over different fields such as $U/K, U/K', \dots$. A variety over K which has an extension over any field containing K is called

⁵ Following Weil [8], we shall say that a variety is defined by a set of equations $f_i(X) = 0$, if the polynomials $f_i(X)$ generate the corresponding prime ideal in $K[X]$.

⁶ See Weil [8], p. 15, Theorem 3. We observe here that the prime ideal determined by the point (x) in $\tilde{K}[X]$ has a basis in $K[X]$ if and only if the prime ideal determined by (x) in $K[X]$ generates a prime ideal in $\tilde{K}[X]$.

absolutely irreducible. If a variety is absolutely irreducible, then any variety equivalent to it is also absolutely irreducible. A complete set of mutually equivalent absolutely irreducible varieties is called an absolute variety, and any variety in the set is to be regarded as a representative of absolute variety over the particular field. If an absolute variety U has a representative U/K over a field K , then it is said to be defined over K . It is known⁷ that among all the fields over which a given absolute variety U is defined, there is a smallest one which is contained in all of them; this smallest field is called the defining field of U . As an absolute variety is uniquely determined by any one of its representatives, there is little danger of confusion in identifying it sometimes with one of the latter. The absolute varieties are of great importance for us, for it is by means of them that we are going to define the concept of a cycle in S_m . Before doing this, we shall introduce the associated form of a variety.

Let U/K be a variety of dimension d in S_m . As we have shown elsewhere,⁸ there is associated with U/K a K -irreducible form $F(Z, Z^{(1)}, \dots, Z^{(d)})$ of the same degree n in each of the $d+1$ sets of indeterminates $(Z), (Z^{(1)}), \dots, (Z^{(d)})$ in S_m with coefficients in K . This form is called the associated form of the variety U/K , and the number n is called the degree of the variety. Conversely, given any K -irreducible form F of this type satisfying a certain set of algebraic conditions, there is a uniquely determined variety U/K of which F is the associated form. It is easily seen that equivalent varieties have the same associated form; the converse of this statement is also true in case the varieties are separably generated or (which is the same thing) the associated form has no multiple factors. A variety is absolutely irreducible if and only if its associated form is absolutely irreducible. It follows then that an absolute variety is uniquely determined by its associated form; this is the reason why the associated forms are particularly suitable for the study of absolute varieties and cycles.

A cycle of dimension d or d -cycle in S_m is an element of the free Abelian group, the generators of which are the set of all absolute varieties of dimension d in S_m . Thus a d -cycle is a finite set of absolute varieties U_j , called its components, to each of which is assigned an integer n_j which is called the multiplicity of U_j in the d -cycle. If the degrees of the varieties U_j are s_j respectively, then the number $n = \sum_j s_j n_j$ is called the degree of the d -cycle.

A cycle is said to be contained in a variety if each of its components is contained in the variety. A cycle is called positive if all the multiplicities n_j

⁷ See Weil [8], pp. 70-1, Theorem 1, Corollary 3.

⁸ Chow and van der Waerden [5], pp. 693-694.

are positive. If F_j are the associated forms of the varieties U_j , respectively, then the form $F = \prod_j F_j^{n_j}$ is called the associated form of the positive cycle.

A positive cycle is said to be rational over a field K if its associated form is rational over K , i. e. the ratios of its coefficients are rational over K ; and if furthermore the associated form is irreducible over K , then the positive cycle is said to be prime rational⁹ over K . Since such a K -irreducible form determines a variety over K and vice versa, it follows that there is a one-to-one correspondence between the varieties over K and the prime rational cycles over K . It is clear that every rational cycle over K can be factorized in a unique way into a sum of prime rational cycles over K corresponding to the factorization of its associated form into a product of irreducible forms over K . Therefore every rational cycle over K can also be regarded as a sum of varieties over K , each with a certain multiplicity; we shall call these varieties the K -component of the cycle.

Let (x) be a generic point of a variety U/K , then the field $K((x))$, which is uniquely determined up to an isomorphism by the variety, is called the field of U/K . Let (ξ) be any point of U/K , then the ring of all the elements of $K((x))$ of the form $f(x)/g(x)$, where $f(x)$ and $g(x)$ are two homogeneous polynomials of (x) of the same degree and $g(\xi) \neq 0$, is called the quotient ring $\mathfrak{U} = Q(U/K, (\xi))$ of U/K at the point (ξ) . It is clear that if (ξ') is a generic specialization of (ξ) over K , then $Q(U/K, (\xi)) = Q(U/K, (\xi'))$, so that the ring \mathfrak{U} depends only on the subvariety W/K in U/K determined by the point (ξ) or (ξ') over K , not on the particular choice of the generic point (ξ) or (ξ') of W/K . Therefore we shall also call the ring \mathfrak{U} the quotient ring of U/K at the subvariety W/K , and shall sometimes write $\mathfrak{U} = Q(U/K, W)$ to indicate its dependence on W/K . The quotient ring \mathfrak{U} is a local ring in the sense of Krull,¹⁰ i. e. a Noetherian ring in which the set of all non-units is an ideal, and its maximal prime ideal \mathfrak{u}

⁹ This is not the "correct" way to define a rational cycle, but is more convenient for our present purpose. The "correct" definition is as follows. A cycle is said to be rational over a field K if it satisfies the following conditions: (1) Each component of the cycle is algebraic over K ; (2) if an (absolute) variety occurs in the cycle, then all the conjugate varieties over K also occur in it with the same multiplicity; (3) the multiplicity of each component is a multiple of its order of inseparability. See Weil [8], Chapter VII, § 6. Our definition differs from this "correct" definition in that the multiplicity of a component will in general be a multiple of a smaller power of the characteristic of K than the order of inseparability of this component over K .

¹⁰ Krull [7]. Our Appendix, On the Extensions of Local Domains, contains a detailed treatment of many of the concepts and results concerning local rings which are used in the text.

consists of all elements $f(x)/g(x)$ such that $f(\xi) = 0$. It is also known¹¹ that the completion \mathfrak{U}^* of the local ring \mathfrak{U} has no nilpotent elements; in other words, the zero ideal in \mathfrak{U}^* is an intersection of a finite number of prime ideals. Let V/K be another variety, and let \mathfrak{B} be the quotient ring $Q(V/K, (\eta))$ of V/K at one of its points (η) ; if W/K is the subvariety in V/K determined by the point (η) over K , then \mathfrak{B} is also the quotient ring $Q(V/K, W')$ of V/K at the subvariety W'/K . We shall say that the varieties U/K and V/K are analytically equivalent¹² at the points (ξ) and (η) respectively, or at the subvarieties W/K and W'/K respectively, if the completions \mathfrak{U}^* and \mathfrak{B}^* of the two local rings \mathfrak{U} and \mathfrak{B} are K -isomorphic. Local properties of a variety at a point or a subvariety which are invariant under analytical equivalences are called locally analytic properties. It is known¹³ that the property of a variety being simple at a point or subvariety in the absolute sense is a locally analytic property.

For an absolute variety U the concept of a quotient ring can only be defined with respect to a given field of definition. However, in order that this concept has a significance independent of the choice of the particular field of definition, it is advisable to restrict ourselves to the case where the subvariety W is also an absolute variety and the field K is a field of definition for both U and W . Then the ring $Q(U/K, W)$ is called the quotient ring of the absolute variety U at the absolute subvariety W over the field K . This includes in particular the case when W has the dimension 0 and hence consists of a point which is rational over K . The advantage for imposing these restrictions regarding W and K lies in the fact that if K' is any field containing K , then the quotient ring $Q(U/K', W)$ is the extension¹⁴ of the quotient ring $Q(U/K, W)$ over K' , so that the ring $Q(U/K', W)$ is uniquely determined by the ring $Q(U/K, W)$ and the field K' . Furthermore, the completion of $Q(U/K', W)$ is the completion of the extension of the completion of $Q(U/K, W)$ over K' , so that the completion of $Q(U/K', W)$ is also uniquely determined by the completion of $Q(U/K, W)$ and the field K' . Let V be another absolute variety and W' be an absolute subvariety in V ; let K_0 be the smallest field over which all the varieties U, V, W, W' are defined. Then the absolute varieties U and V are said to be analytically

¹¹ See Chevalley [3], p. 11, Theorem 1.

¹² In this definition of analytical equivalence we are following Zariski [10], p. 49, Definition 4.

¹³ See Zariski [10], pp. 49-51, Theorem 15, where this theorem is proved for the most general case.

¹⁴ See Appendix.

equivalent at the subvarieties W and W' respectively, if the completions of the two local rings $Q(U/K_0, W)$ and $Q(V/K_0, W')$ are K_0 -isomorphic. It follows then from what we have just said above that if K is any field containing K_0 , then the completions of the two local rings $Q(U/K, W)$ and $Q(V/K, W')$ are also K -isomorphic; or, in other words, the (relative) varieties U/K and V/K are analytically equivalent at the subvarieties W/K and W'/K respectively.

Let U/K and V/K be two varieties in the space S_m and S_t respectively, and let (x) be a generic point of U/K . If V/K has a generic point (y) which is rational over $K((x))$, then the correspondence $(x) \rightarrow (y)$ defines a rational transformation T of U/K onto V/K ; and the pair $(x), (y)$ is said to be a pair of corresponding generic points of U/K and V/K respectively under the rational transformation T , and we shall write $(y) = T(x)$. For every point (ξ) of U/K , its image under the rational transformation T consists of all the points (η) such that (ξ, η) is a specialization of (x, y) over the specialization $(x) \rightarrow (\xi)$ over K . If for a suitably chosen linear form $\sum_{j=0}^t a_j y_j$, the elements $y_i / \sum_{j=0}^t a_j y_j$, $i = 0, 1, \dots, t$, are all in the quotient ring $Q(U/K, (\xi))$, then the transformation T is said to be regular at the point (ξ) . In this case the image of (ξ) under T is a uniquely determined point of V/K . The rational transformation T is called birational, if conversely the point (x) is also rational over $K((y))$, i. e. if we have $K((x)) = K((y))$; the resulting rational transformation of V/K onto U/K is called the inverse T^{-1} of the transformation T . If a birational transformation T is regular at a point (ξ) and the inverse transformation T^{-1} is also regular at the image point (η) of (ξ) under T , then the transformation is said to be biregular at this point (ξ) or at the pair of points $(\xi), (\eta)$. Two varieties U/K and V/K which are in biregular correspondence at the two points (ξ) and (η) respectively, have evidently the same quotient rings $Q(U/K, (\xi))$ and $Q(V/K, (\eta))$; hence they are also analytically equivalent at these two points. These concepts can be carried over to the case of two absolute varieties, just as we have done above for the definitions of quotient ring and analytical equivalence; we have then to replace the points (ξ) and (η) by two absolute subvarieties W and W' in U and V respectively, and the field K must be a field of definition for all the varieties U, V, W, W' .

2. Statement of the problem and result. Let U/K be a variety of r dimensions in the projective space S_m . As we have indicated at the beginning of the previous section, any positive d -cycle of degree n in U/K can be repre-

sented through its associated form by a point in the projective space S_t , and the set of all such points in S_t constitutes exactly the points of a bunch of varieties over K . Let V/K be any variety contained in this bunch, then the set of all the positive cycles in U/K corresponding to the points in V/K is called an algebraic system of positive cycles in U/K , and the variety V/K is called the associated variety of the algebraic system. Let (y) be a generic point of V/K , then the cycle $G(y)$ determined by (y) is called a generic cycle of the algebraic system and we shall denote the algebraic system itself by $|G(y)|$. We shall assume that the generic cycle $G(y)$ is prime rational over the field $K((y))$, and that a generic point (x) of $G(y)$ over $K((y))$ is also a generic point of U/K . This last assumption means simply that U/K is the carrier variety of the algebraic system $|G(y)|$. The pair of points $(x), (y)$ then determines an algebraic correspondence T between the varieties U/K and V/K , which we shall call the associated correspondence of the algebraic system $|G(y)|$.

The problem with which we are concerned in this paper can be described as follows. Let \bar{T} be any (K -irreducible) algebraic correspondence between the variety U/K and another variety \bar{V}/K (which may of course lie in a different space). If (\bar{y}) is a generic point of \bar{V}/K , then the set of all points in U/K which correspond to (\bar{y}) under the correspondence \bar{T} constitutes a variety or prime rational cycle $G(\bar{y})$ in U/K over the field $K((\bar{y}))$. If this cycle $G(\bar{y})$ has the dimension d and the degree n , then there corresponds to $G(\bar{y})$ by means of its natural coordinates a point (y) in the projective space S_t ; let V/K be the variety in S_t determined by the generic point (y) over the field K . This variety V/K then determines an algebraic system of positive cycles in U/K , of which the cycle $G(y) = G(\bar{y})$ is a generic member, and there is an algebraic correspondence T between U/K and V/K associated with this algebraic system. Thus every algebraic correspondence \bar{T} between U/K and a variety \bar{V}/K induces in U/K an algebraic system of positive cycles, but in general the variety \bar{V}/K is of course different from the associated variety V/K of the algebraic system and the correspondence \bar{T} is different from the associated correspondence T . In fact, under the correspondence \bar{T} between U/K and \bar{V}/K , the points of \bar{V}/K will not in general be in one-to-one correspondence with the positive cycles of the algebraic system without exception; that is, to some points of \bar{V}/K there might correspond more than one or even an infinite number of cycles of the algebraic system.¹⁵ The question arises as to whether there are any other properties which distinguish

¹⁵ For a more detailed description of the relations between the correspondences T and \bar{T} , see Chow and van der Waerden [5], § 2.

the associated variety V/K from the rather arbitrarily chosen variety \bar{V}/K , apart from the fact that V/K represents in a one-to-one manner the cycles of the algebraic system $|G(y)|$. One would naturally ask for those properties which are invariant under a biregular birational transformation of the associated variety V/K . In terms of local geometry we can formulate the question as follows: Let $(\bar{\eta})$ be a point of \bar{V}/K to which there corresponds a uniquely determined cycle $G(\bar{\eta})$ in U/K and let (η) be the corresponding point in V/K so that we have $G(\eta) = G(\bar{\eta})$. Then both varieties V/K and \bar{V}/K can be said to represent through the correspondences T and \bar{T} respectively the algebraic system of positive cycles $|G(y)| = |G(\bar{y})|$ in a one-to-one manner in the neighborhoods of the points (η) and $(\bar{\eta})$ respectively. The question is whether there are any locally birational or analytic properties which distinguish the associated correspondence T in the neighborhood of the pair (η) , $G(\eta)$ from the correspondence \bar{T} in the neighborhood of the pair $(\bar{\eta})$, $G(\bar{\eta})$. In this paper we shall give an answer to this question in the special case where the correspondence \bar{T} is a rational transformation of U/K onto \bar{V}/K and consequently the associated correspondence T is also a rational transformation of U/K onto V/K . We shall show that in this case the local analytic properties of the variety V/K in the neighborhood of the point (η) has a very simple connection with the local properties of the variety U/K in the neighborhood of the cycle $G(\eta)$, which is not true in general for the variety \bar{V}/K in the neighborhood of $(\bar{\eta})$.

We shall from now on restrict ourselves to the case where the associated correspondence T of the algebraic system $|G(y)|$ is a rational transformation of U/K onto V/K . It is clear in this case that the variety V/K has the dimension $r - d$. We shall call such an algebraic system of positive cycles an involutorial system, on account of the fact that in case $d = 0$ such a system is essentially what is usually called an involution in the classical algebraic geometry. For the sake of convenience, we shall assume that the ground field K contains infinitely many elements. Our main result is the following theorem:

THEOREM. *Let U/K be a variety; let $|G(y)|$ be an involutorial system of positive cycles in U/K with the associated variety V/K and the associated transformation T , where (y) is a generic point of V/K . Let (η) be a rational point of V/K and (ξ) be a generic point of a K -component G'/K of $G(\eta)$; we assume that the field $K((\xi))$ is analytically disjoint¹⁰ with respect to the quotient ring $Q(V/K, (\eta))$ over K ; this implies in particular that*

¹⁰ For the definition of analytical disjointness, see Appendix.

$K((y))$ and $K((\xi))$ are linearly disjoint over K , so that the product variety W/K of V/K and G'/K is defined. If the transformation T is regular at the point (ξ) and the variety G'/K is a simple K -component of $G(\eta)$ and is separably generated, then the varieties U/K and W/K are analytically equivalent at the points (ξ) and (η, ξ) respectively.

For the sake of greater generality, we have stated our theorem above for the "relative" varieties; this is the reason why we have to make the assumptions that the point (η) is rational and the field $K((\xi))$ is analytically disjoint with respect to the quotient ring $Q(V/K, (\eta))$ over K . In the applications, however, we are mostly concerned with the case where both varieties U/K and V/K are absolutely irreducible; in other words, we are concerned with an involutorial system of positive cycles in an absolute variety U with an absolute variety V as the associated variety. It is then natural to assume that G' is an absolute variety, i. e. an (absolute) component of the positive cycle $G(\eta)$, and that the field K is the smallest field over which all the varieties $U, V, G', (\eta)$ are defined. Since the field $K((\xi))$ is in this case a regular extension of K , it follows (Appendix, Theorem 4) that $K((\xi))$ is analytically disjoint with respect to any local domain with K as a basic field, hence in particular with respect to the local domain $Q(V/K, (\eta))$ over K . Recalling our definition of the analytical equivalence between two absolute varieties, we can state our theorem for absolute varieties as follows:

THEOREM. *Let U be an absolute variety; let $|G(y)|$ be an involutorial system of positive cycles in U with the associated (absolute) variety V and the associated transformation T . Let (η) be any point of V and let G' be a component of $G(\eta)$. If G' is a simple component of $G(\eta)$ and the transformation T is regular at the subvariety G' , then the variety U and the product variety $W = V \times G'$ are analytically equivalent at the subvarieties G' and $(\eta) \times G'$ respectively.*

From this theorem we can derive as an immediate corollary a criterion for simple points or subvarieties. Since the product variety W is simple at the subvariety $(\eta) \times G'$ if and only if both V and G' are simple at the subvarieties (η) and G' respectively, and since the variety G' is certainly simple at the variety G' itself, i. e. at its generic point, it follows that under the conditions of the above theorem the variety U is simple at the subvariety G' if and only if the variety V is simple at the point (η) . It is evident that the application of this criterion to the problem of Jacobian variety gives an affirmative answer to the question we have posed at the beginning of the previous section.

Finally, we should like to add that the significance of our theorem goes beyond a mere criterion for simple points; it gives us a certain information about the analytic structure of U in the neighborhood of a cycle of an algebraic system. In order to express this more clearly, we introduce yet another definition. An involutorial system $|G(y)|$ in an absolute variety U is called a fiber system if the transformation T is regular at every point of U and the cycle $G(\eta)$ is an absolute variety (i. e. consisting of one component with multiplicity 1) for every point (η) of V . It is evident that in this case each point of U belongs to exactly one cycle of the system, so that the entire variety U is fibered by the algebraic system of cycles. Our result can then be expressed roughly as follows: If $|G(y)|$ is a fiber system in U with the associated variety V and the associated transformation T , then U has properties similar to those of a fiber space in the topology, with V as the base space and T as the projection. In fact, there seems to be more than a formal analogy between this algebraic "fiber space" and the fiber space in topology; we hope to be able to come back to these questions on some future occasions.

Before we proceed with the proof of our theorem, we shall make a change in our terminology. It is clear that our main theorem is of a purely local nature, and as such it can be stated just as well for varieties in affine spaces instead of projective spaces. In fact, this is the form in which we shall prove the theorem in the next two sections; for, as is usually the case with a problem of purely local nature, it is very convenient to be able to work with the affine coordinates. It is easily seen that most of the definitions and concepts as developed and indicated in this and the previous sections can be carried over with suitable modifications to the varieties and points in affine spaces, and we can replace such field extensions as $K((x))$ by the ordinary field extension $K(x)$. Thus we shall from now on consider only varieties and points in a suitably chosen affine space in the projective space S_m ; we shall denote this affine space also by S_m and represent a point in it by the affine coordinates $(x) = (x_1, \dots, x_m)$. We shall assume that the affine S_m has been so chosen that the d -cycle $G(\eta)$ is finite; this implies of course that the variety U/K as well as the generic cycle $G(y)$ of the system are also finite. A variety is said to be finite with respect to an affine space if every generic point of it is finite (i. e. in the affine space), and a cycle is said to be finite if all its components are finite. Similarly, we can assume that the affine S_t has been so chosen that the point (η) is finite, which implies that the variety V/K is also finite; but for the sake of convenience later in our proof, we shall make a more definite choice of this affine space S_t with respect to a given choice of the affine coordinate system in the affine space S_m .

We set $\omega_0 = (Z_0 Z^{(1)}_1 \cdots Z^{(d)}_d)^n$ in the ordered set of monomials $\omega_0, \omega_1, \dots, \omega_t$ in the $d+1$ set of indeterminates $(Z), (Z^{(1)}), \dots, (Z^{(d)})$ as defined in the beginning of the previous section. Then the affine S_t is defined as the set of all points (c_0, c_1, \dots, c_t) with $c_0 \neq 0$, and such a point will be denoted by its affine coordinates $(c_1/c_0, \dots, c_t/c_0)$. In order to insure that the point (η) is finite with respect to such a choice of the affine S_t , we shall assume that the affine coordinate system in the affine space S_m has been so chosen that the linear variety defined by the d equations $X_i = v_i, i = 1, \dots, d$, where the v_1, \dots, v_d are independent variables over K , intersects the d -cycle $G(\eta)$ properly and finitely, i.e. the intersection consists of a finite number of finite points. Then the coefficient of the term ω_0 in the associated form $F_{(\eta)}(Z, Z^{(1)}, \dots, Z^{(d)})$ of $G(\eta)$ cannot be zero. For, let $v^{(i)}_0 = -v_i, v^{(i)}_i = 1, v^{(i)}_j = 0$, for $i = 1, \dots, d$, and $j \neq i$, then the form $F_{(\eta)}(Z, v^{(1)}, \dots, v^{(d)})$ is the associated form of the intersection of $G(\eta)$ with the linear variety. It is easily seen that the coefficient of the term Z_0^n in the form $F_{(\eta)}(Z, v^{(1)}, \dots, v^{(d)})$ is exactly the coefficient of the term ω_0 in the form $F_{(\eta)}(Z, Z^{(1)}, \dots, Z^{(d)})$. If this coefficient were zero, then the form $F_{(\eta)}(Z, v^{(1)}, \dots, v^{(d)})$ must contain as factor at least one linear form without the term Z_0 , and this linear form would then correspond to an infinite intersection point of $G(\eta)$ with the linear variety, in contradiction to our assumption. Thus the coefficient of the term ω_0 in the form $F_{(\eta)}(Z, Z^{(1)}, \dots, Z^{(d)})$ is not zero, and this means that the point (η) is in the affine space S_t . From now on we shall assume that the associated form has been so normalized that the coefficient of the term ω_0 is equal to 1; the coefficients of the rest of the terms will then be the affine coordinates of the corresponding cycle.

3. Two lemmas. We begin by recalling some well known facts concerning the semi-local rings. A ring \mathfrak{o} is called a semi-local ring,¹⁷ if it is a Noetherian ring and contains only a finite number of maximal prime ideals. We shall consider \mathfrak{o} also as a topological space in the following way. Since \mathfrak{o} is in particular a group with respect to the addition, we can define a topology in \mathfrak{o} by specifying a fundamental system of neighborhoods of the element 0. Let \mathfrak{a} be the product of all the maximal prime ideals of \mathfrak{o} , then the sets \mathfrak{a}^s , for $s = 1, 2, \dots$, will be taken as a fundamental system of neighborhoods of 0. It can be shown that the usual conditions are satisfied; they are all trivial except the one asserting that $\bigcap_{s=1}^{\infty} \mathfrak{a}^s = 0$, which is the well known

¹⁷ Chevalley [1].

theorem due to Krull. Thus \mathfrak{o} is a topological group and consequently a regular space; moreover, not only the addition but also the multiplication is a continuous function in \mathfrak{o} .

Let \mathfrak{o}' be another semi-local ring which contains \mathfrak{o} as a sub-ring, and let \mathfrak{o}' be topologized in a similar way by means of the product α' of its maximal prime ideals. The question arises as to the relation between the topologies of the two rings \mathfrak{o} and \mathfrak{o}' . In general, they are not concordant; in other words, the sets $\alpha'^s \cap \mathfrak{o}$ do not in general constitute a fundamental system of neighborhoods of 0 in \mathfrak{o} . However, in the important special case when \mathfrak{o}' is a finite \mathfrak{o} -module (and this implies itself that \mathfrak{o}' is a semi-local ring) and no non-zero element of \mathfrak{o} is a zero-divisor in \mathfrak{o}' , it can be shown¹⁸ that the sets $\alpha'^s \cap \mathfrak{o}$ also constitute a fundamental system of neighborhoods of 0 in \mathfrak{o} , so that \mathfrak{o}' contains \mathfrak{o} not only as a sub-ring, but also as a sub-space. Explicitly this means that, in this case, there exist two sequences of positive integers $l(s)$ and $m(s)$, with $l(s) \rightarrow \infty$ and $m(s) \rightarrow \infty$ as $s \rightarrow \infty$, such that the relations $\alpha^{l(s)} \subset \alpha'^s \cap \mathfrak{o} \subset \alpha^{m(s)}$ hold for all s .

Let U/K and V/K be two varieties of the same dimension r in the affine spaces S_m and S_t respectively. Let T be a rational transformation of U/K onto V/K , and let (x) and $(y) = T(x)$ be a pair of corresponding generic points of U/K and V/K respectively. Then $K(x)$ is a finite algebraic extension of $K(y)$; let n be the degree of the extension. To the generic point (y) of V/K there correspond in the inverse transformation T^{-1} n (not necessarily distinct) points $(x^{(1)}) = (x), (x^{(2)}), \dots, (x^{(n)})$ of U/K , which are all generic points of U/K and which together constitute a complete set of conjugates over $K(y)$. We shall say that the inverse transformation T^{-1} is defined at a point (η) of V/K , if there is a finite specialization $(x^{(1)}, \dots, x^{(n)}) \rightarrow (\xi^{(1)}, \dots, \xi^{(n)})$ over the specialization $(y) \rightarrow (\eta)$, such that every specialization of $(x^{(1)}, \dots, x^{(n)})$ over the specialization $(y) \rightarrow (\eta)$ coincides with this one except possibly for a reordering of the points $(\xi^{(1)}), \dots, (\xi^{(n)})$. It is clear that if the rational transformation T is one which is associated with an involutorial system of positive 0-cycles on U/K , so that V/K is the associated variety of the system, then the inverse transformation T^{-1} is defined at every point of the variety V/K , for a suitable choice of the affine spaces S_m and S_t .

LEMMA I. *Let T be a rational transformation of a variety U/K onto a variety V/K , both of the same dimension r . Let (ξ) be a point of U/K*

¹⁸ See Chevalley [1], p. 699, Proposition 7. For the properties of semi-local rings used in the following, we refer once for all to this paper of Chevalley.

such that T is regular at (ξ) and the inverse transformation T^{-1} is defined at the image point $(\eta) = T(\xi)$. Then the quotient ring $\mathfrak{U} = Q(U/K, (\xi))$ contains the quotient ring $\mathfrak{B} = Q(V/K, (\eta))$ as a subspace.

Proof. Let $r = K[y]$ be the coordinate ring of V/K , and let \mathfrak{p} be the prime ideal in r determined by the point (η) , and let $\mathfrak{s} = r - \mathfrak{p}$ be the multiplicatively closed system of all the elements of r not in \mathfrak{p} . Then the quotient ring of r with respect to \mathfrak{s} is a local ring \mathfrak{o} , and the ideal $\mathfrak{m} = \mathfrak{p}$ is the maximal prime ideal of \mathfrak{o} . It is easily seen that \mathfrak{o} is the quotient ring $\mathfrak{B} = Q(V/K, (\eta))$ of V/K at (η) . Let $r' = K[x]$ be the coordinate ring of U/K , and let $\mathfrak{p}'_1, \dots, \mathfrak{p}'_n$ be the n prime ideals (not necessarily distinct) in r' determined by the n points $(\xi^{(1)}), \dots, (\xi^{(n)})$, and let $\mathfrak{s}' = \bigcap_{i=1}^n (r' - \mathfrak{p}'_i)$ be the multiplicatively closed system of all elements of r' not in any one of the ideals $\mathfrak{p}'_1, \dots, \mathfrak{p}'_n$. Let \mathfrak{o}' be the quotient ring of r' with respect to \mathfrak{s}' . Then \mathfrak{o}' is a semi-local ring, and its maximal prime ideals are the ideals $\mathfrak{m}'_i = \mathfrak{o}'\mathfrak{p}'_i$ ($i = 1, \dots, n$), which are not necessarily distinct. Let \mathfrak{o}'_i be the quotient ring of \mathfrak{o}' with respect to the multiplicatively closed system $\mathfrak{o}' - \mathfrak{m}'_i$; each ring \mathfrak{o}'_i is a local ring with the maximal prime ideal $\mathfrak{o}'_i\mathfrak{m}'_i$. It is clear that \mathfrak{o}'_1 is the quotient ring $\mathfrak{U} = Q(U/K, (\xi))$ of U/K at (ξ) . We have to show that if $\mathfrak{o}'_1 \supset \mathfrak{o}$, then \mathfrak{o} is a sub-space of \mathfrak{o}'_1 . We shall achieve this by constructing a bigger ring which contains both \mathfrak{o}'_1 and \mathfrak{o} as sub-spaces.

Consider the ring $\mathfrak{R} = K[x^{(1)}, \dots, x^{(n)}]$, and for each i ($i = 1, \dots, n$), let \mathfrak{P}_{ij} , $j = 1, \dots, a$, be the distinct prime ideals determined by the specializations of $(x^{(1)}, \dots, x^{(n)})$ over the specialization $(x^{(i)}) \rightarrow (\xi^{(i)})$ over K . Since there is an automorphism of the field $k(x^{(1)}, \dots, x^{(n)})$ over $K(y)$ which carries any one of the points $(x^{(i)})$ into any other, it follows that we can arrange the ideals \mathfrak{P}_{ij} in such a way that the ideals $\mathfrak{P}_{1j}, \dots, \mathfrak{P}_{nj}$ are conjugate to each other for every $j = 1, 2, \dots, a$. Let $\mathfrak{S} = \bigcap_{i,j} (\mathfrak{R} - \mathfrak{P}_{ij})$ be the multiplicatively closed system of all elements of \mathfrak{R} not contained in any one of the prime ideals \mathfrak{P}_{ij} , and let \mathfrak{D} be the quotient ring of \mathfrak{R} with respect to \mathfrak{S} . The ring \mathfrak{D} is then a semi-local ring, and its maximal prime ideals are the ideals $\mathfrak{M}_{ij} = \mathfrak{D}\mathfrak{P}_{ij}$. Let $\tilde{\mathfrak{D}}$ be the integral closure of \mathfrak{D} ; since $\tilde{\mathfrak{D}}$ is a finite \mathfrak{D} -module, it is also a semi-local ring. For every i and j , let $\tilde{\mathfrak{M}}_{ijk}$ ($k = 1, \dots, b$) be the distinct maximal prime ideals of $\tilde{\mathfrak{D}}$ such that $\tilde{\mathfrak{M}}_{ijk} \cap \mathfrak{D} = \mathfrak{M}_{ij}$; and we shall arrange these ideals in such a way that the ideals $\tilde{\mathfrak{M}}_{1jk}, \dots, \tilde{\mathfrak{M}}_{njk}$ are conjugate to each other, for every j and k . Now, since every element of \mathfrak{o} is finite over any finite specialization of the elements of \mathfrak{D} , it is integrally dependent on \mathfrak{D} ; hence, we have $\tilde{\mathfrak{D}} \supset \mathfrak{o}$. On the other hand, since every element of $\tilde{\mathfrak{D}}$ is also finite over any finite specialization of

the elements of \mathfrak{o} and is consequently integrally dependent on \mathfrak{o} , we conclude that $\bar{\mathfrak{D}}$ is the integral closure of \mathfrak{o} in the field $K(x^{(1)}, \dots, x^{(n)})$. From this it follows that $\bar{\mathfrak{D}}$ is a finite \mathfrak{o} -module, and hence $\bar{\mathfrak{D}}$ contains \mathfrak{o} as a sub-space, and we have $\bar{\mathfrak{M}}_{ijk} \cap \mathfrak{o} = \mathfrak{m}$.

It is clear from our construction that $\mathfrak{R} \supset \mathfrak{r}'$; and, as all the prime ideals \mathfrak{P}_{ij} of \mathfrak{R} contract in \mathfrak{r}' to one of ideals $\mathfrak{p}'_1, \dots, \mathfrak{p}'_n$, we have $\mathfrak{r}' \cap \mathfrak{S} \subset \mathfrak{s}'$, from which it follows that $\mathfrak{D} \supset \mathfrak{o}'$. The ideals \mathfrak{P}_{ij} ($j=1, \dots, a$) are evidently the only maximal prime ideals of \mathfrak{R} which contract to \mathfrak{p}'_1 in \mathfrak{r}' . It follows that the ideals \mathfrak{M}_{ij} ($j=1, \dots, a$) consist of all the maximal prime ideals of \mathfrak{D} which contract to \mathfrak{m}'_1 in \mathfrak{o}' . Hence the ideals $\bar{\mathfrak{M}}_{ijk}$ are all the maximal prime ideals of $\bar{\mathfrak{D}}$ which contract to \mathfrak{m}'_1 to \mathfrak{o}' . Let $\bar{\mathfrak{D}}_1$ be the quotient ring of $\bar{\mathfrak{D}}$ with respect to the multiplicatively closed system $\bigcap_{j,k} (\bar{\mathfrak{D}} - \bar{\mathfrak{M}}_{ijk})$. Since $\bar{\mathfrak{M}}_{ijk} \cap \mathfrak{o}' = \mathfrak{m}'_1$, the ring \mathfrak{o}'_1 is contained in $\bar{\mathfrak{D}}_1$. On the other hand, it is easily seen that every element of $\bar{\mathfrak{D}}_1$ is finite over any finite specialization of the elements of \mathfrak{o}'_1 and hence is integrally dependent on \mathfrak{o}'_1 . Therefore, $\bar{\mathfrak{D}}_1$ is the integral closure of \mathfrak{o}'_1 in the field $K(x^{(1)}, \dots, x^{(n)})$, from which it follows that $\bar{\mathfrak{D}}_1$ is a finite \mathfrak{o}'_1 -modul and consequently contains \mathfrak{o}'_1 as a subspace.

We shall now show that $\bar{\mathfrak{D}}_1$ contains also \mathfrak{o} as a sub-space. Let $\mathfrak{M}_i = \prod_{j,k} \bar{\mathfrak{M}}_{ijk}$ and $\mathfrak{M}' = \prod' \bar{\mathfrak{M}}_{ijk}$, where the product \prod' extends over all the distinct ideals only among the $\bar{\mathfrak{M}}_{ijk}$. Since $\bar{\mathfrak{D}}$ contains \mathfrak{o} as a sub-space, and since the sets \mathfrak{M}^s ($s=1, 2, \dots$) constitute by definition a fundamental system of neighborhoods of 0 in $\bar{\mathfrak{D}}$, we have the relations $\alpha^{l(s)} \subset \mathfrak{M}^s \cap \mathfrak{o} \subset \alpha^{m(s)}$ with $l(s) \rightarrow \infty$ and $m(s) \rightarrow \infty$, as $s \rightarrow \infty$. Since any two distinct ones of the ideals $\bar{\mathfrak{M}}_{ijk}^s$ are relatively prime to each other, we have

$$\begin{aligned} \mathfrak{M}^s &= \prod'_{i,j,k} \bar{\mathfrak{M}}_{ijk}^s = \bigcap_{i,j,k} \bar{\mathfrak{M}}_{ijk}^s = \bigcap_i \left(\bigcap_{j,k} \bar{\mathfrak{M}}_{ijk}^s \right) = \bigcap_i \left(\prod_{j,k} \bar{\mathfrak{M}}_{ijk}^s \right) \\ &= \bigcap_i \mathfrak{M}_i^s. \end{aligned}$$

Hence we have $\alpha^{l(s)} \subset \bigcap_i (\mathfrak{M}_i^s \cap \mathfrak{o}) \subset \alpha^{m(s)}$. Now, since the ideals \mathfrak{M}_i^s are all conjugate to each other over $K(y)$, we have evidently $\mathfrak{M}_1^s \cap \mathfrak{o} = \dots = \mathfrak{M}_n^s \cap \mathfrak{o}$. Hence we have the relations $\alpha^{l(s)} \subset \mathfrak{M}_1^s \cap \mathfrak{o} \subset \alpha^{m(s)}$, which means that the sets $\mathfrak{M}_1^s \cap \mathfrak{o}$ constitute a fundamental system of neighborhoods of 0 in \mathfrak{o} . Now, the sets $(\bar{\mathfrak{D}}_1 \mathfrak{M}_1)^s$ constitute by definition a fundamental system of neighborhoods of 0 in $\bar{\mathfrak{D}}_1$, and we have from the theory of quotient rings¹⁹

¹⁹ We observe that since the $\bar{\mathfrak{M}}_{ijk}$ are maximal prime ideals in $\bar{\mathfrak{D}}$, we have $\mathfrak{M}_1^s = \bigcap_{j,k} \bar{\mathfrak{M}}_{ijk}^s$, and each $\bar{\mathfrak{M}}_{ijk}^s$ is a primary ideal. Therefore we have $\bar{\mathfrak{D}}_1 \bar{\mathfrak{M}}_{ijk}^s \cap \bar{\mathfrak{D}} = \bar{\mathfrak{M}}_{ijk}^s$ and hence also $\bar{\mathfrak{D}}_1 \mathfrak{M}_1^s \cap \bar{\mathfrak{D}} = \mathfrak{M}_1^s$.

that $(\bar{\mathfrak{D}}_1 \mathfrak{A}_1)^* \cap \bar{\mathfrak{D}} = \mathfrak{A}_1^*$. Therefore we have $(\bar{\mathfrak{D}}_1 \mathfrak{A}_1)^* \cap \mathfrak{o} = (\bar{\mathfrak{D}}_1 \mathfrak{A}_1)^* \cap \bar{\mathfrak{D}} \cap \mathfrak{o} = \mathfrak{A}_1^* \cap \mathfrak{o}$, which shows that \mathfrak{o} is a sub-space of $\bar{\mathfrak{D}}_1$.

Since the ring $\bar{\mathfrak{D}}_1$ contains both \mathfrak{o}'_1 and \mathfrak{o} as sub-spaces, and since (on account of the regularity of the rational transformation T at $(\xi^{(1)})$) \mathfrak{o}'_1 contains \mathfrak{o} , it follows that \mathfrak{o}'_1 contains \mathfrak{o} as a sub-space. Thus the lemma is proved.

[Note added in proof (March 10, 1950): Professor Zariski has kindly called our attention to the fact that our Lemma I is closely related to a result in his recent note in the *Proceedings of the National Academy of Sciences*, vol. 35 (1949), pp. 62-66, Theorem 3. In fact, if the variety V/K is analytically irreducible at the point (η) , our Lemma I is an immediate consequence of Zariski's theorem. In case V/K is not analytically irreducible, it is also possible to deduce our Lemma I from Zariski's theorem, if we make use of Zariski's theory of normal varieties. This can be done in the following way, as communicated to us by Professor Zariski.

Let \bar{U}/K and \bar{V}/K be derived normal models of U/K and V/K respectively. Let $(\bar{\xi}^{(1)}), (\bar{\xi}^{(2)}), \dots, (\bar{\xi}^{(g)})$ be the points of \bar{U}/K which correspond to (ξ) and let similarly $(\bar{\eta}^{(1)}), (\bar{\eta}^{(2)}), \dots, (\bar{\eta}^{(h)})$ be the points of \bar{V}/K which correspond to (η) . Since (η) is the only point of V/K which corresponds to (ξ) under T , the points of \bar{V}/K which correspond to any $(\bar{\xi}^{(i)})$ (in the rational transformation \bar{T} of \bar{U}/K onto \bar{V}/K) are among the points $(\bar{\eta}^{(j)})$ and hence are finite in number. Since \bar{U}/K is normal it follows that to each point $(\bar{\xi}^{(i)})$ there corresponds a unique point, say $(\bar{\eta}^{(\mu_i)})$, and that \bar{T} is regular at $(\bar{\xi}^{(i)})$. Recalling the fact that a normal variety is analytically irreducible at any one of its points (see Zariski, *Annals of Mathematics*, vol. 49 (1948), pp. 352-361), it follows from Zariski's theorem that

(1) the quotient ring of $(\bar{\eta}^{(\mu_i)})$ is a sub-space of the quotient ring of $(\bar{\xi}^{(i)})$ ($i = 1, 2, \dots, g$).

Now let us assume that in every specialization $(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \rightarrow (\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(n)})$ over the specialization $(y) \rightarrow (\eta)$ the point (ξ) occurs at least once among the $(\xi^{(i)})$ (this assumption is weaker than the assumption that the inverse transformation T^{-1} is defined at (η)). We have for each $j = 1, 2, \dots, h$ the specialization $(y, \bar{y}) \rightarrow (\eta, \bar{\eta}^{(j)})$ (where (\bar{y}) is the point \bar{V}/K which corresponds to the general point (y) of V/K), and if we apply the above assumption to the specializations of $(x^{(1)}, x^{(2)}, \dots, x^{(n)})$ which are over the specialization $(y, \bar{y}) \rightarrow (\eta, \bar{\eta}^{(j)})$ we conclude at once that

(2) each of the h points $(\bar{\eta}^{(j)})$ occurs at least once among the points $(\bar{\eta}^{(\mu_i)})$, $i = 1, 2, \dots, g$.

From (1) and (2) it follows that the intersection of the quotient rings $\mathfrak{o}(\bar{\eta}^{(j)})$ of the h points $(\bar{\eta}^{(j)})$ is a sub-space of the intersection of the quotient rings $\mathfrak{o}'(\bar{\xi}^{(i)})$ of the g points $(\bar{\xi}^{(i)})$. Since \bar{V}/K is a derived normal model of V/K , the quotient ring \mathfrak{o} of the point (η) is a sub-space of $\bigcap_j \mathfrak{o}(\bar{\eta}^{(j)})$. Hence \mathfrak{o} is a sub-space of $\bigcap_i \mathfrak{o}'(\bar{\xi}^{(i)})$. Since also \bar{U}/K is a derived normal model of U/K , the quotient ring \mathfrak{o}'_1 of the point $(\xi) = (\xi^{(1)})$ is a sub-space of $\bigcap_i \mathfrak{o}'(\bar{\xi}^{(i)})$. From these conclusions and from the fact that the space \mathfrak{o} is a subset of the space \mathfrak{o}'_1 (since T is regular at (ξ)), our Lemma I follows.]

Before we proceed to the second lemma, we shall make a few remarks about the coefficients of the associated form. Let U/K be a variety in S_m . Consider an involutorial system of positive 0-cycles of degree n in U/K with the associated variety V/K and the associated transformation T , and let (x) and $y = T(x)$ be a pair of corresponding generic points of U/K and V/K respectively. Let $F_{(y)}(Z)$ be the associated form of the generic 0-cycle $G(y)$ of the system. We have then

$$F_{(y)}(Z) = \prod_{i=1}^n (Z_0 + \sum_{j=1}^m Z_j x^{(i)}_j),$$

where the n points $(x^{(1)}) = (x)$, $(x^{(2)})$, \dots , $(x^{(n)})$ form a complete set of conjugates over $K(y)$. The coordinates of the point (y) are the coefficients of the form $F_{(y)}(Z)$, and they are by hypothesis rational functions of (x) . Let $f_k(Z_0, Z_k)$ be the form obtained from $F_{(y)}(Z)$ by setting $Z_j = 0$ for all $j \neq 0, k$; we have then evidently

$$f_k(Z_0, Z_k) = \prod_{i=1}^n (Z_0 + Z_k x^{(i)}_k), \quad k = 1, \dots, m.$$

Thus, for each $k = 1, \dots, m$, the coefficients of the form $f_k(Z_0, Z_k)$ are the elementary symmetric functions of the n elements $x^{(1)}_k, \dots, x^{(n)}_k$. It is clear that the coefficients of the m forms $f_k(Z_0, Z_k)$, $k = 1, \dots, m$, are all part of the coefficients of the form $F_{(y)}(Z)$; that is, they all occur among the coordinates of the point (y) . Furthermore, let

$$g_k(Z_0, Z_k) = \prod_{i=2}^n (Z_0 + Z_k x^{(i)}_k), \quad k = 1, \dots, m$$

so that we have $f_k(Z_0, Z_k) = (Z_0 + Z_k x^{(1)}_k) g_k(Z_0, Z_k)$, $k = 1, \dots, m$. It is well known that for each $k = 1, \dots, m$, the coefficients of $g_k(Z_0, Z_k)$, being the elementary symmetric functions of the $n-1$ elements $x^{(2)}_k, \dots, x^{(n)}_k$, can be expressed as polynomials in the coefficients of the form $f_k(Z_0, Z_k)$ and

the element x_k . It follows then that the coefficients of the forms $g_k(Z_0, Z_k)$, $k = 1, \dots, m$, are all elements of the ring $K[x, y]$. We shall make use of this remark presently.

LEMMA II. *Let U/K be a variety; let $|G(y)|$ be an involutory system of positive 0-cycles in U/K with the associated variety V/K and the associated transformation T . Let (ξ) be a point of U/K such that the transformation T is regular at (ξ) , and that the corresponding point $(\eta) = T(\xi)$ on V/K determines an 0-cycle $G(\eta)$ which contains (ξ) as a simple component. Then the maximal prime ideal \mathfrak{u} of the quotient ring $\mathfrak{U} = Q(U/K, (\xi))$ has a basis consisting of elements in the quotient ring $\mathfrak{B} = Q(V/K, (\eta))$.*

Proof. Let (x) and $(y) = T(x)$ be a pair of corresponding generic points of U/K and V/K respectively. The hypothesis that the point (ξ) is a simple component of the special cycle $G(\eta)$ implies that the generic cycle $G(y)$ can only have simple components; hence the field $K(x)$ must be a separable algebraic extension of $K(y)$ and the n points $(x^{(1)}) = (x)$, $(x^{(2)})$, \dots , $(x^{(n)})$ of the generic cycle $G(y)$ are distinct. Let $(x^{(1)}, \dots, x^{(n)}) \rightarrow (\xi^{(1)}, \dots, \xi^{(n)})$ be any specialization over the specialization $(x) \rightarrow (\xi)$ over K ; the points $(\xi^{(1)}), \dots, (\xi^{(n)})$ are then evidently the n points of the cycle $G(\eta)$. Since the point (ξ) occurs only once in the cycle $G(\eta)$, we have $(\xi^{(i)}) \neq (\xi)$ for all $i \neq 1$. Therefore, we can assume that the affine coordinate system in S_m has been so chosen that we have $\xi^{(i)}_j \neq \xi_j$ for all $i \neq 1$ and all j . It is easily seen that this condition can always be obtained by means of a suitably chosen affine transformation with coefficients in K , the field K being assumed to contain infinitely many elements. Let $\lambda = (i_1, \dots, i_m)$ be any ordered set of m positive integers not greater than n , we shall set $(\xi^\lambda) = (\xi^{(i_1)}_{i_1}, \dots, \xi^{(i_m)}_{i_m})$. Then, as λ runs through all the n^m possible ordered sets (i_1, \dots, i_m) of positive integers not greater than n , we obtain a set of n^m (not necessarily distinct) points $\{(\xi^\lambda)\}$. Since we have $\xi^{(i)}_j \neq \xi_j$ for all $i \neq 1$ and all j , it is clear that the point (ξ) occurs only once among the n^m points of the set (ξ^λ) . Similarly, we set $(x^\lambda) = (x^{(i_1)}_{i_1}, \dots, x^{(i_m)}_{i_m})$ and obtain a set of n^m points $\{(x^\lambda)\}$. It is clear that for every λ the specialization $(x^\lambda) \rightarrow (\xi^\lambda)$ is the only specialization over the specialization $(x^{(1)}, \dots, x^{(n)}) \rightarrow (\xi^{(1)}, \dots, \xi^{(n)})$ over K .

Let $\psi_j(X)$ ($j = 1, \dots, h$) be a basis of the prime ideal determined by the point (ξ) in the polynomial ring $K[X] = K[X_1, \dots, X_m]$. It is easily seen that the h elements $\psi_j(x)$, $j = 1, \dots, h$, constitute then also a basis of the ideal \mathfrak{u} in the local ring \mathfrak{U} . Since the point (ξ) occurs only once among the set $\{(\xi^\lambda)\}$, it follows that for every $\lambda \neq (1, \dots, 1)$ the h elements

$\psi_j(\xi^\lambda)$, $j = 1, \dots, h$, are not all zero. There exist then h linearly independent combinations

$$\phi_j(X) = \sum_{k=1}^h a_{jk} \psi_k(X), \quad j = 1, \dots, h,$$

of the polynomials $\psi_j(X)$ with coefficients a_{jk} in K , such that for every $\lambda \neq (1, \dots, 1)$ none of the h elements $\phi_j(\xi^\lambda)$, $j = 1, \dots, h$, vanishes. The elements $\phi_j(x)$, $j = 1, \dots, h$, constitute of course also a basis of the ideal \mathfrak{u} in \mathfrak{U} ; and we have for each λ and j the unique specialization $\phi_j(x^\lambda) \rightarrow \phi_j(\xi^\lambda)$ over the specialization $(x^{(1)}, \dots, x^{(n)}) \rightarrow (\xi^{(1)}, \dots, \xi^{(n)})$ over K .

Now we set

$$\Phi_j = \prod_{\lambda} \phi_j(x^\lambda), \quad j = 1, \dots, h,$$

where the product runs over all the n^m points of the set $\{(x^\lambda)\}$. We prove first that these h elements Φ_j , $j = 1, \dots, h$, are elements of the ring $K[y]$; this evidently implies that they are also elements of the quotient ring \mathfrak{B} . To show this, we set $\phi_j(X) = \phi_j^{(0)}(X_1, \dots, X_m)$, $j = 1, \dots, h$; we can then write

$$\Phi_j = \prod_{i_1=1}^n \dots \prod_{i_m=1}^n \phi_j^{(0)}(x^{(i_1)}_1, \dots, x^{(i_m)}_m), \quad j = 1, \dots, h.$$

Consider the polynomials

$$\phi_j^{(1)}(X_2, \dots, X_m) = \prod_{i_1=1}^n \phi_j^{(0)}(x^{(i_1)}_1, X_2, \dots, X_m), \quad j = 1, \dots, h;$$

we have evidently

$$\Phi_j = \prod_{i_1=1}^n \dots \prod_{i_2=1}^n \phi_j^{(1)}(x^{(i_2)}_2, \dots, x^{(i_m)}_m), \quad j = 1, \dots, h.$$

Since the coefficients of the polynomials $\phi_j^{(1)}(X_2, \dots, X_m)$, $j = 1, \dots, h$, are integral rational symmetric functions of the n elements $x^{(1)}_2, \dots, x^{(n)}_2$ they can be expressed as integral rational functions of the elementary symmetric functions of these n elements. This means that the coefficients of the polynomials $\phi_j^{(1)}(X_2, \dots, X_m)$, $j = 1, \dots, h$, are integral rational functions of the coefficients of the form $f_1(Z_0, Z_1)$; and since the coefficients of $f_1(Z_0, Z_1)$ are themselves some of the coordinates of the point (y) , it follows that the coefficients of $\phi_j^{(1)}(X_2, \dots, X_m)$, $j = 1, \dots, h$, are elements of $K[y]$. Consider next the polynomials

$$\phi_j^{(2)}(X_3, \dots, X_m) = \prod_{i_2=1}^n \phi_j^{(1)}(x^{(i_2)}_2, X_3, \dots, X_m), \quad j = 1, \dots, h;$$

we have evidently

$$\Phi_j = \prod_{i_1=1}^n \cdots \prod_{i_m=1}^n \phi_j^{(2)}(x^{(i_1)}_1, \cdots, x^{(i_m)}_m), \quad j = 1, \cdots, h.$$

Since the coefficients of the polynomials $\phi_j^{(2)}(X_1, \cdots, X_m)$, $j = 1, \cdots, h$, are integral rational symmetric functions of the n elements $x^{(1)}_1, \cdots, x^{(n)}_1$ with coefficients in $K[y]$, it follows that they can be expressed as integral rational functions of the elementary symmetric functions of these n elements also with coefficients in $K[y]$. This means that the coefficients of the polynomials $\phi_j^{(2)}(X_1, \cdots, X_m)$, $j = 1, \cdots, h$, are integral rational functions of the coefficients of the form $f_2(Z_0, Z_2)$, with coefficients in $K[y]$; and since the coefficients of the form $f_2(Z_0, Z_2)$ are themselves some of the coordinates of the point (y) , it follows that the coefficients of $\phi_j^{(2)}(X_1, \cdots, X_m)$, $j = 1, \cdots, h$, are elements of $K[y]$. Thus, proceeding successively in this manner, we conclude after m steps that the elements Φ_j , $j = 1, \cdots, h$, are elements of $K[y]$.

We maintain that the h elements Φ_j , $j = 1, \cdots, h$, constitute a basis of the ideal u in the ring \mathfrak{U} . Since we have $\Phi_j = (\Phi_j/\phi_j(x))\phi_j(x)$, $j = 1, \cdots, h$, and since the h elements $\phi_j(x)$, $j = 1, \cdots, h$, constitute a basis of u in \mathfrak{U} , it is sufficient to prove that the elements $\Phi_j/\phi_j(x)$, $j = 1, \cdots, h$, are units of the ring \mathfrak{U} . We first show that these elements are elements of the ring \mathfrak{U} .

In fact, indicating by $\prod'_{(k)}^{(m)}$ the product over all the $m - k + 1$ indices i_k, \cdots, i_m with the exception of the one combination $i_k = \cdots = i_m = 1$, we obtain by a simple calculation the following:

$$\begin{aligned} \Phi_j/\phi_j(x) &= \prod'_{(1)}^{(m)} \phi_j^{(0)}(x^{(i_1)}_1, \cdots, x^{(i_m)}_m) \\ &= \prod_{i_1=2}^n \phi_j^{(0)}(x^{(i_1)}_1, x_2, \cdots, x_m) \prod'_{(2)}^{(m)} \phi_j^{(1)}(x^{(i_2)}_2, \cdots, x^{(i_m)}_m) \\ &= \cdots \\ &= \prod_{k=1}^m \left(\prod_{i_k=2}^n \phi_j^{(k-1)}(x^{(i_k)}_k, x_{k+1}, \cdots, x_m) \right) = \prod_{k=1}^m \theta_j^{(k)}(x_{k+1}, \cdots, x_m), \\ &\quad (j = 1, \cdots, h), \end{aligned}$$

where in the last step we have set

$$\theta_j^{(k)}(X_{k+1}, \cdots, X_m) = \prod_{i_k=2}^n \phi_j^{(k-1)}(x^{(i_k)}_k, X_{k+1}, \cdots, X_m);$$

$$k = 1, \cdots, m; j = 1, \cdots, h.$$

Thus it is only necessary to prove that for each $k = 1, \cdots, m$ and each $j = 1, \cdots, h$, the element $\theta_j^{(k)}(x_{k+1}, \cdots, x_m)$ is an element of \mathfrak{U} . Since the coefficients of the polynomial $\theta_j^{(k)}(X_{k+1}, \cdots, X_m)$ are integral rational symmetric functions of the $n - 1$ elements $x^{(2)}_k, \cdots, x^{(n)}_k$ with coefficients

in $K[y]$, they can be expressed as integral rational functions of the elementary symmetric functions of these $n - 1$ elements, also with coefficients in $K[y]$. This means that the coefficients of $\theta_j^{(k)}(X_{k+1}, \dots, X_m)$ are integral rational functions of the coefficients of the form $g_k(Z_0, Z_k)$, with coefficients in $K[y]$; and since the coefficients of $g_k(Z_0, Z_k)$ are themselves elements of $K[x, y]$, it follows that the coefficients of $\theta_j^{(k)}(X_{k+1}, \dots, X_m)$ are also elements of $K[x, y]$. Therefore, the element $\theta_j^{(k)}(x_{k+1}, \dots, x_m)$ is also an element of $K[x, y]$; and since the transformation T is regular at the point (ξ) , the element $\theta_j^{(k)}(x_{k+1}, \dots, x_m)$ is in the quotient ring \mathfrak{U} . Thus we have shown that the elements $\Phi_j/\phi_j(x)$, $j = 1, \dots, h$, are all elements of the ring \mathfrak{U} .

Finally, we show that the elements $\Phi_j/\phi_j(x)$, $j = 1, \dots, h$, are units of the ring \mathfrak{U} . In fact, we have

$$\Phi_j/\phi_j(x) = \prod_{\lambda}' \phi_j(x^{\lambda}), \quad j = 1, \dots, h,$$

where the product \prod_{λ}' runs over all the n^m points of the set $\{(x^{\lambda})\}$ except the point (x) . For every λ and j , we have the unique specialization $\phi_j(x^{\lambda}) \rightarrow \phi_j(\xi^{\lambda})$ over the specialization $(x^{(1)}, \dots, x^{(n)}) \rightarrow (\xi^{(1)}, \dots, \xi^{(n)})$ over K ; hence we have also the unique specialization

$$\Phi_j/\phi_j(x) = \prod_{\lambda}' \phi_j(x^{\lambda}) \rightarrow \prod_{\lambda}' \phi_j(\xi^{\lambda}), \quad j = 1, \dots, h,$$

over the specialization $(x^{(1)}, \dots, x^{(n)}) \rightarrow (\xi^{(1)}, \dots, \xi^{(n)})$ over K . Since the elements $\Phi_j/\phi_j(x)$, $j = 1, \dots, h$, are elements of \mathfrak{U} , it follows that the specialization

$$\Phi_j/\phi_j(x) \rightarrow \prod_{\lambda}' \phi_j(\xi^{\lambda}), \quad j = 1, \dots, h,$$

is the uniquely determined specialization of these h elements over the specialization $(x) \rightarrow (\xi)$ over K . Now, we have shown above that $\phi_j(\xi^{\lambda}) \neq 0$ for every $\lambda \neq (1, \dots, 1)$ and every j ; hence we have also

$$\prod_{\lambda}' \phi_j(\xi^{\lambda}) \neq 0, \quad j = 1, \dots, h.$$

This means that the h elements $\Phi_j/\phi_j(x)$, $j = 1, \dots, h$, are not in \mathfrak{u} and hence units of \mathfrak{U} . This concludes the proof of Lemma II.

4. Proof of the theorem. We restate here our theorem with slight modifications for affine varieties:

THEOREM. Let U/K be a variety; let $|G(y)|$ be an involutorial system of positive cycles in U/K with the associated variety V/K and the associated transformation T , where (y) is a generic point of V/K . Let (η) be a

rational point of V/K such that the cycle $G(\eta)$ is finite, and let (ξ) be a generic point of a K -component G'/K of $G(\eta)$. We assume that the field $K(\xi)$ is analytically disjoint with respect to the ring $Q(V/K, (\eta))$ over K ; this implies in particular that the fields $K(\xi)$ and $K(\eta)$ are linearly disjoint over K , so that the product variety $W/K = V/K \times G'/K$ is defined. If the transformation T is regular at the point (ξ) and if the variety G'/K is a simple K -component of $G(\eta)$ and is separably generated, then the varieties U/K and W/K are analytically equivalent at the points (ξ) and (η, ξ) respectively.

Proof. As before, let (x) and $(y) = T(x)$ be a pair of corresponding generic points of U/K and V/K respectively. We shall first prove the theorem for the case when the dimension d of the positive cycles of the involutorial system is zero and then extend it by a well known device to the general case.

(a) The case $d = 0$. The assumption that G'/K is a simple K -component of $G(\eta)$ and is separably generated implies that the point (ξ) is a simple component of the 0-cycle $G(\eta)$; hence the hypotheses of both Lemma I and Lemma II are satisfied. Let $\mathfrak{U} = Q(U/K, (\xi))$ be the quotient ring of U/K at the point (ξ) and \mathfrak{u} be the maximal prime ideal in \mathfrak{U} ; let $\mathfrak{B} = Q(V/K, (\eta))$ be the quotient ring of V/K at the point (η) ; and let $\mathfrak{B}^* = (W/K, (\eta, \xi))$ be the quotient ring of W/K at the point (η, ξ) . Let \mathfrak{U}^* be the completion of the local ring \mathfrak{U} and let \mathfrak{u}^* be its maximal prime ideal; let \mathfrak{B}^* be the completion of the local ring \mathfrak{B} and \mathfrak{b}^* be its maximal prime ideal; let \mathfrak{B}^* be the completion of the local ring \mathfrak{B} . Our aim is to prove that the two rings \mathfrak{U}^* and \mathfrak{B}^* are K -isomorphic. We observe first that \mathfrak{B} is also the quotient ring $Q(V/K(\xi), (\eta))$ of the extension variety $V/K(\xi)$ of V/K over $K(\xi)$ at the point (η) ; which means that \mathfrak{B} is the extension of \mathfrak{B} over the field $K(\xi)$. Furthermore, since $K(\xi)$ is a finite extension of K and since \mathfrak{B} contains K as a coefficient field, it follows that $\mathfrak{B}^* = K(\xi) \times \mathfrak{B}^*$. According to Lemma I, \mathfrak{U} contains \mathfrak{B} as a sub-space; hence \mathfrak{U}^* contains \mathfrak{B}^* as a subring. Since $K(\xi)$ is separable over K , \mathfrak{U} contains K as a basic field; hence the completion \mathfrak{U}^* contains a coefficient field K' over K , which is K -isomorphic to the residue ring $\mathfrak{U}^*/\mathfrak{u}^*$ and hence also K -isomorphic to $K(\xi)$. The field K' is not in general a subfield of the universal domain \mathfrak{Q} and is hence an "abstract" field in our terminology; however, since K' is K -isomorphic with the field $K(\xi)$, we can without loss of generality identify the two fields by setting $K' = K(\xi)$ through the given K -isomorphism. If we now set

$R = \mathfrak{U}^*$, $\mathfrak{o} = \mathfrak{B}$, $L = K'$ in the Theorem 7 of the Appendix, it is easily seen that all the conditions in the hypothesis are fulfilled; we can therefore conclude that the subring $K'\mathfrak{B}^*$ in \mathfrak{U}^* is an (K', \mathfrak{B}^*) -isomorphic image of $\mathfrak{B}^* = K' \times \mathfrak{B}^*$. This shows in particular that $K'\mathfrak{B}^*$ is a complete local ring and its maximal prime ideal is the ideal $K'\mathfrak{b}^*$, which is contained in the ideal \mathfrak{u}^* ; it follows then that the subring $K'\mathfrak{B}^*$ is equal to its own adherence in \mathfrak{U}^* . Therefore, to complete our proof, we only need to show that $\mathfrak{U}^* = K'\mathfrak{B}^*$; for then the two rings \mathfrak{U}^* and \mathfrak{B}^* will be K' -isomorphic and *a fortiori* also K -isomorphic. To prove this equation $\mathfrak{U}^* = K'\mathfrak{B}^*$, it is sufficient to show that the elements of $K'\mathfrak{B}$ are everywhere dense in the local ring \mathfrak{U}^* . This means that given any element α of \mathfrak{U}^* and a positive integer s , there is an element β_s of $K'\mathfrak{B}$ such that $\alpha \equiv \beta_s \pmod{\mathfrak{u}^{*s}}$. Now, according to Lemma II, the ideal \mathfrak{u} of \mathfrak{U} has a basis Φ_j , $j = 1, \dots, h$, which are elements of \mathfrak{B} , hence also of $K'\mathfrak{B}$. It is well known that the elements Φ_j , $j = 1, \dots, h$, constitute then also a basis of the ideal \mathfrak{u}^* in \mathfrak{U}^* . Since the elements of K' constitute a complete set of representatives of the residue field $\mathfrak{U}^*/\mathfrak{u}^*$, there exists an element α_1 in K' such that $\alpha \equiv \alpha_1 \pmod{\mathfrak{u}^*}$. We set $\beta_1 = \alpha_1$. Assuming that the element β_{s-1} has already been constructed, we proceed to construct the element β_s . Since the element $\alpha - \beta_{s-1}$ is an element in the ideal \mathfrak{u}^{*s-1} , we can express it as a form of degree $s-1$ in the basis elements Φ_j , $j = 1, \dots, h$, with coefficients in \mathfrak{U}^* . Now, by the case $s=1$, each of these coefficients is congruent to an element of K' modulo \mathfrak{u}^* . If we now substitute every coefficient of the form by the corresponding element in K' , we obtain an element α_s in $K'\mathfrak{B}$ such that $\alpha - \beta_{s-1} \equiv \alpha_s \pmod{\mathfrak{u}^{*s}}$. The element $\beta_s = \alpha_s + \beta_{s-1}$ is in the ring $K'\mathfrak{B}$ and satisfies the condition $\alpha \equiv \beta_s \pmod{\mathfrak{u}^{*s}}$. Thus the proof for the case $d=0$ is complete.

(b) The case $d > 0$. As we have mentioned at the end of Section 2, we shall assume that the affine coordinate system of S_m has been so chosen that the $(m-d)$ -dimensional linear variety defined by the equations $X_j = v_j$, $j = 1, \dots, d$, where the v_1, \dots, v_d are independent variables over K , intersects the d -cycle $G(\eta)$ properly and finitely. Then, if G_1/K is any K -component of degree n_1 of $G(\eta)$, a generic point (ξ) of G_1/K can be chosen such that the first d coordinates ξ_1, \dots, ξ_d are any given set of d independent variables over K , and the field $K(\xi_1, \dots, \xi_m)$ is an algebraic extension of degree n_1 over the field $K(\xi_1, \dots, \xi_d)$. Since the first d coordinates x_1, \dots, x_d of the generic point (x) of U/K are in this case independent variables over K , we can therefore set $\xi_j = x_j$ for $j = 1, \dots, d$. In other words, each K -component of $G(\eta)$ has a generic point with the first d coordinates x_1, \dots, x_d . This holds in particular for the K -component G'/K

of $G(\eta)$, and we have a generic point (ξ) with $\xi_j = x_j$ for $j = 1, \dots, d$. Since the variety G'/K is separably generated, we may without loss of generality assume that the affine coordinate system of S_m has been so chosen that $K(\xi)$ is separably algebraic of degree n' over field $K(x_1, \dots, x_d)$.

Let $\tilde{K} = K(x_1, \dots, x_d)$, and let U/\tilde{K} and V/\tilde{K} be the extensions of the varieties U/K and V/K respectively over \tilde{K} . Let L/\tilde{K} be the linear variety of dimension $m - d$ defined by the d linear equations $X_j = x_j$, $j = 1, \dots, d$. Then the intersection of U/\tilde{K} with L/\tilde{K} is a variety \tilde{U}/\tilde{K} over \tilde{K} , and the point (x) is also a generic point of \tilde{U}/\tilde{K} . The point (ξ) is evidently also a point of \tilde{U}/\tilde{K} , and it is well known²⁰ that the quotient ring $Q(\tilde{U}/\tilde{K}, (\xi))$ coincides with the quotient ring $\mathfrak{U} = Q(U/K, (\xi))$. Since the point (x) is a generic point of \tilde{U}/\tilde{K} and the point (y) is also a point of V/\tilde{K} , the rational transformation $(y) = T(x)$ of U/K onto V/K can also be regarded as a rational transformation \tilde{T} of the variety \tilde{U}/\tilde{K} into the variety V/\tilde{K} . In fact, this rational transformation T is also a transformation of \tilde{U}/\tilde{K} onto V/\tilde{K} ; in other words, the point (y) is not only a generic point of V/\tilde{K} , but also a generic point of V/\tilde{K} . To prove this, we only need to show that every generic point (y') of V/\tilde{K} is a specialization of the point (y) over \tilde{K} . In fact, since $G(y')$ is a generic d -cycle of the involutory system $|G(y)|$, it has at least a finite intersection point $(x') = (x_1, \dots, x_d, x'_{d+1}, \dots, x'_m)$ with the linear variety L/\tilde{K} . The pair of points (x') , (y') is then a specialization of the pair (x) , (y) over \tilde{K} ; hence, the pair (x_1, \dots, x_d) , (y') is also a specialization of the pair (x_1, \dots, x_d) , (y) over \tilde{K} , and this means that the point (y') is a specialization of (y) over \tilde{K} .

It is evident from the definition of the transformation \tilde{T} that to the point (y) of V/\tilde{K} correspond in the inverse transformation \tilde{T}^{-1} exactly the points of the intersection of L/\tilde{K} with the d -cycle $G(y)$. Since each K -component G_1/K of $G(\eta)$ has a generic point of the type $(\xi) = (x_1, \dots, x_d, \xi_{d+1}, \dots, \xi_m)$ such that the field $K(\xi)$ is an algebraic extension of degree n_1 over \tilde{K} , it follows that the intersection of G_1/K (considered as a d -cycle) with L/\tilde{K} is a finite 0-cycle \tilde{G}_1 of degree n_1 , prime rational over \tilde{K} ; thus we have a 0-dimensional variety \tilde{G}_1/\tilde{K} , and the point (ξ) is also a generic point of \tilde{G}_1/\tilde{K} . This holds in particular for the intersection \tilde{G}'/\tilde{K} of G'/\tilde{K} with L/\tilde{K} ; and since the field $\tilde{K}(\xi) = K(\xi)$ is separable over \tilde{K} , this 0-dimensional variety \tilde{G}'/\tilde{K} is also separably generated over \tilde{K} . Summing over all the K -components of the cycle $G(\eta)$, we conclude that the intersection of $G(\eta)$ with L/\tilde{K} is a finite 0-cycle $\tilde{G}(\eta)$ of degree n , rational over \tilde{K} , which

²⁰ See Zariski [10], p. 8, Lemma 1.

contains the variety \tilde{G}'/\tilde{K} as a simple \tilde{K} -component. It is well known that in such a case the intersection of the generic cycle $G(y)$ with L/\tilde{K} is also a finite 0-cycle $\tilde{G}(y)$ of degree n , prime rational over $\tilde{K}(y)$, and the cycle $\tilde{G}(\eta)$ is the unique specialization of the cycle $\tilde{G}(y)$ over the specialization $(y) \rightarrow (\eta)$ over \tilde{K} . Thus the rational transformation \tilde{T} induces in \tilde{U}/\tilde{K} an involutorial system of 0-cycles $|\tilde{G}(y)|$, and the inverse transformation \tilde{T}^{-1} is defined at the point (η) . Since the transformation \tilde{T} is evidently regular at the point (ξ) , it follows then from Lemma I that the quotient ring $\mathfrak{U} = Q(\tilde{U}/\tilde{K}, (\xi))$ contains the quotient ring $\tilde{\mathfrak{B}} = Q(V/\tilde{K}, (\eta))$ as a sub-space.

The variety V/\tilde{K} is however not the associated variety of the involutorial system $|\tilde{G}(y)|$. To find the associated variety of this system, we only need to observe that the associated form of the generic cycle $\tilde{G}(y)$ is the form $\tilde{F}(Z) = F_{(y)}(Z, u^{(1)}, \dots, u^{(d)})$, where the form $F_{(y)}(Z, Z^{(1)}, \dots, Z^{(d)})$ is the associated form of the d -cycle $G(y)$ and the $u^{(i)}$, $i = 1, \dots, d$, are defined as follows: For each $i = 1, \dots, d$, we set $u^{(i)}_j = -x_i$, 1, or 0, according as $j = 0$, $j = i$, or $j \neq 0, i$, respectively. As we have mentioned at the end of Section 2, the form $F_{(y)}(Z, Z^{(1)}, \dots, Z^{(d)})$ is assumed to be so normalized that the coefficient of the term ω_0 is 1. It follows then the coefficient of the term Z_0^n in the form $\tilde{F}(Z)$ is also equal to 1, so that $\tilde{F}(Z)$ is already in the normalized form. Let (\tilde{y}) be the set of coefficients (except the coefficient 1 of the term Z_0^n) of this form $\tilde{F}(Z)$, ordered in some arbitrary but fixed way, then the variety \tilde{V}/\tilde{K} determined by the point (\tilde{y}) over \tilde{K} is the associated variety of the involutorial system $|\tilde{G}(y)|$; and we can now write $\tilde{G}(\tilde{y})$ instead of $\tilde{G}(y)$ to indicate its dependence on the point (\tilde{y}) . It is clear that the coordinates of (\tilde{y}) are linear combination of the coordinates of (y) ; this means that there is a projection P of V/\tilde{K} onto \tilde{V}/\tilde{K} and that this projection P is regular at every finite point of V/\tilde{K} , hence in particular at the point (η) . We shall denote by $(\tilde{\eta})$ the corresponding point of (η) under this projection P , and write $\tilde{G}(\tilde{\eta})$ instead of $\tilde{G}(\eta)$ to indicate its dependence on the point $(\tilde{\eta})$ of \tilde{V}/\tilde{K} . Since the hypothesis of Lemma I is evidently satisfied by P at the point (η) , it follows that the quotient ring $\tilde{\mathfrak{B}} = Q(V/\tilde{K}, (\eta))$ contains the quotient ring $\tilde{\mathfrak{B}} = Q(\tilde{V}/\tilde{K}, (\tilde{\eta}))$ of \tilde{V}/\tilde{K} at the point $(\tilde{\eta})$ as a sub-space; and this means that $\tilde{\mathfrak{B}}$ is also a sub-space of the local ring \mathfrak{U} .

Let \tilde{T} be the associated rational transformation of the involutorial system $|\tilde{G}(\tilde{y})|$; it is easily seen that \tilde{T} is the product of the rational transformation \tilde{T} of \tilde{U}/\tilde{K} onto V/\tilde{K} and the projection P of V/\tilde{K} onto \tilde{V}/\tilde{K} . Since the transformation \tilde{T} is regular at the point (ξ) and the image point on V/\tilde{K} is the point (η) , and since the projection P is regular at the point (η) and the

image point on \tilde{V}/\tilde{K} is the point $(\tilde{\eta})$, it follows that the associated transformation \tilde{T} is regular at the point (ξ) and the image point on \tilde{V}/\tilde{K} is the point $(\tilde{\eta})$. Moreover, the 0-dimensional variety \tilde{G}'/\tilde{K} is a simple \tilde{K} -component of the cycle $\tilde{G}(\tilde{\eta})$ and is separably generated; hence the hypothesis of Lemma II is fulfilled by the involutorial system $|\tilde{G}(\tilde{y})|$ on the variety \tilde{U}/\tilde{K} with the associated variety \tilde{V}/\tilde{K} and the associated transformation \tilde{T} . It follows then that the maximal prime ideal \mathfrak{u} in the quotient ring $\mathfrak{U} = Q(\tilde{U}/\tilde{K}, (\xi))$ has a basis consisting of elements in the subring \mathfrak{B} and hence also in the subring \mathfrak{B} of \mathfrak{U} .

Finally, let $\mathfrak{B} = Q(W/K, (\eta, \xi))$ be the quotient ring of W/K at the point (η, ξ) . It is clear that \mathfrak{B} is also the quotient ring $Q(V/K(\xi), (\eta))$ of the extension variety $V/K(\xi)$ of V/K over $K(\xi)$ at the point (η) , and the variety $V/K(\xi)$ is also the extension variety of V/\tilde{K} over $K(\xi)$; this means that the local ring \mathfrak{B} is the extension of the local ring \mathfrak{B} over the field $K(\xi)$. Since $K(\xi)$ is a finite extension of \tilde{K} and since \tilde{V} contains \tilde{K} as a coefficient field, it follows that $\mathfrak{B}^* = K(\xi) \times \mathfrak{B}^*$. Furthermore, since the field $K(\xi)$ is analytically disjoint with respect to the quotient ring $\mathfrak{B} = Q(V/K, (\eta))$ over the field K , and since \mathfrak{B} is the extension of \mathfrak{B} over the subfield \tilde{K} of $K(\xi)$, it follows (Appendix, Theorem 6) that $K(\xi)$ is also analytically disjoint with respect to \mathfrak{B} over the field \tilde{K} .

We are now ready to apply the method of (a). Consider the completion \mathfrak{U}^* of \mathfrak{U} ; it contains the completion \mathfrak{B}^* of \mathfrak{B} as a subring. Since $K(\xi)$ is separable over \tilde{K} , the local ring \mathfrak{U} contains \tilde{K} as a basic field; hence the completion \mathfrak{U}^* contains a coefficient field \tilde{K}' over \tilde{K} , which is \tilde{K} -isomorphic to the field $K(\xi)$. Here again the field \tilde{K}' is an "abstract" field, but we can just as before identify it with $K(\xi)$ by setting $\tilde{K}' = K(\xi)$ through the given \tilde{K} -isomorphism. If we set $R = \mathfrak{U}^*$, $\mathfrak{o} = \mathfrak{B}$, $L = \tilde{K}'$, $K = \tilde{K}$ in Theorem 7 of the Appendix, then all the condition in the hypothesis are fulfilled; we can therefore conclude that the subring $\tilde{K}'\mathfrak{B}^*$ in \mathfrak{U}^* is a $(\tilde{K}', \mathfrak{B}^*)$ -isomorphic image of $\mathfrak{B}^* = \tilde{K}' \times \mathfrak{B}^*$. Observing that the maximal prime ideal in \mathfrak{U}^* has a basis consisting of elements in \mathfrak{B} , we can repeat the same argument as in the last part of (a) and conclude that $\mathfrak{U}^* = \tilde{K}'\mathfrak{B}^*$. Therefore the two rings \mathfrak{U}^* and \mathfrak{B}^* are \tilde{K}' -isomorphic and *a fortiori* also K -isomorphic with each other. This completes the proof of our theorem.

APPENDIX.*

On the Extensions of Local Domains.

In this appendix we shall develop some notions and results concerning the extension of a local domain over a field, which are used in the text, but are not directly related to the problem considered there, so that we consider it more appropriate to present them separately. Theorems 1-6 are not essentially new; in fact, most of them are contained implicitly in the work of Chevalley, to which we shall refer for some of the proofs. However, for our present purpose, we have to develop the subject in a somewhat different way. The one new result is Theorem 7, which plays an essential part in the proof of the main theorem in the text. As a matter of terminology, we stress here that the expression "field" will be used in this appendix in its usual general sense as meaning an abstract field, not necessarily a subfield of a certain "universal domain."

Let R_1 and R_2 be two (commutative) rings which contain a field K as common subring. We shall consider the Kronecker product $R_1 \times R_2$ of R_1 and R_2 over the field K , which is a ring containing both R_1 and R_2 as subrings. The Kronecker product $R_1 \times R_2$ has the following characteristic property: Let R' be any ring containing K , and let R'_1 and R'_2 be two subrings of R' , both containing K ; then if R'_1 and R'_2 are K -isomorphic to R_1 and R_2 respectively, then the subring $^{21} R'_1 R'_2$ in R' is a K -homomorphic image of $R_1 \times R_2$. Furthermore, this homomorphism will be an isomorphism if and only if R'_1 and R'_2 are linearly disjoint over K . We recall here that R'_1 and R'_2 are linearly disjoint over K , if every set of linearly independent elements of R'_1 over K is still such over R'_2 ; and that when this is so, then every set of linearly independent elements of R'_2 over K is also still such over R'_1 . If we identify the rings R'_1 and R'_2 with the rings R_1 and R_2 respectively, as we shall often do, then the homomorphism of $R_1 \times R_2$ onto the subring $R_1 R_2$ in R' is not only a K -homomorphism, but also an (R_1, R_2) -homomorphism in the sense that both R_1 and R_2 are left invariant by the homomorphism.

Let L and M be two fields which contain a common subfield K . We shall say that L and M are *algebraically disjoint* over K , if the prime ideal in the polynomial ring $K[X_1, \dots, X_n]$ determined by every set of n elements

* Added January 20, 1950.

²¹ We shall denote by $R'_1 R'_2$ the subring of R' consisting of all the sums of products of elements in R'_1 and R'_2 . Thus, in case of two subfields L and M of a larger field, the ring LM is in general *not* the compositum of L and M .

in L generates a prime ideal in the polynomial ring $M[X_1, \dots, X_n]$. It is easily seen that in case both L and M are subfields of a larger field such that L and M are independent over K , then they are algebraically disjoint over K if and only if they are linearly disjoint over K . On the other hand, given any two fields L and M having exactly a common subfield K , there always exists a field N which contains L and a subfield M' , K -isomorphic to M , such that L and M' are independent over K . If L and M are algebraically disjoint over K , then L and M' are also algebraically disjoint over K and hence also linearly disjoint over K ; and since in this case the intersection of L and M' is exactly the field K , we can identify M' with M , so that N contains both L and M as subfields. Thus there is no essential difference between algebraical disjointness and linear disjointness; the former is a natural generalization of the latter to two arbitrary fields over K and hence more suitable to use in those investigations in which it is not possible or convenient to impose a fixed "universal domain" in advance.

The above described relation between algebraical disjointness and the linear disjointness enables us to carry over most of the results about the linear disjointness to algebraical disjointness. Thus, in particular, we can conclude that algebraical disjointness is a symmetrical relation between the two fields. Furthermore, we can define a field L as a regular extension of K , if L and \bar{K} (the algebraical closure of K) are algebraically disjoint over K . It follows then from a well known result²² that if L is a regular extension of K and M is any field containing K , then L and M are algebraically disjoint over K .

The definition of algebraical disjointness can be extended to two integral domains. We shall say that two integral domains R_1 and R_2 , both containing the field K as subring, are algebraically disjoint over K , if their corresponding fields of quotients L and M are algebraically disjoint over K .

THEOREM 1. *Let R_1 and R_2 be two integral domains which both contain the field K as subring. Then the Kronecker product $R_1 \times R_2$ of R_1 and R_2 over K is an integral domain if and only if R_1 and R_2 are algebraically disjoint over K .*

Proof. If $R_1 \times R_2$ is an integral domain, then the fields L and M will be subfields in the field of quotients of $R_1 \times R_2$, and since R_1 and R_2 are linearly disjoint over K , it follows that L and M are also linearly disjoint over K and hence algebraically disjoint over K . Conversely, if L and M are algebraically disjoint over K , let N be a field containing both L and M as

²² See Weil [8], p. 18, Theorem 5.

subfields, such that L and M are linearly disjoint over K . Then the subring LM in N is a homomorphic image of $L \times M$; and since L and M are linearly disjoint over K , this homomorphism is an isomorphism. Since LM is an integral domain, it follows that $L \times M$ is an integral domain, and hence $R_1 \times R_2$ is also an integral domain.

THEOREM 2. *Let R be an integral domain containing the field K as subring, and let L be an algebraic extension of K . If L and R are algebraically disjoint over K , then any R -homomorphism of $L \times R$ is an isomorphism.*

Proof. In fact, if M is the field of quotients of R , then any R -homomorphism of $L \times R$ can be extended to an M -homomorphism of the ring of quotients of $L \times R$ with respect to the multiplicatively closed system R , which is the ring $L \times M$. Since L is algebraic, $L \times M$ is a field, and hence the homomorphism must be an isomorphism.

Following Cohen,²³ we shall define a generalized local ring \mathfrak{o} as a (commutative) ring with an element 1 in which (1) the set \mathfrak{m} of all non-units is an ideal with finite basis, and (2) $\bigcap_{s=1}^{\infty} \mathfrak{m}^s = (0)$. If a generalized local ring is a Noetherian ring, then it is a local ring in the sense of Krull. It is known²⁴ that a complete generalized local ring is a local ring, so that in any case the completion \mathfrak{o}^* of \mathfrak{o} is a local ring. We shall speak of a local domain or generalized local domain if the ring has no zero-divisors.

If a generalized local ring \mathfrak{o} contains a field K as subring, then no element of K is contained in the maximal prime ideal \mathfrak{m} of \mathfrak{o} , and hence the residue ring $\mathfrak{o}/\mathfrak{m}$ can be considered as an extension field of K . Furthermore, the completion \mathfrak{o}^* of \mathfrak{o} also contains the field K as subring, and the residue ring $\mathfrak{o}^*/\mathfrak{o}^*\mathfrak{m}$, which is K -isomorphic to $\mathfrak{o}/\mathfrak{m}$, can also be considered as an extension field of K . Any field K' in the ring \mathfrak{o} which is a complete system of representatives of the residue ring $\mathfrak{o}/\mathfrak{m}$ is called a *coefficient field* in \mathfrak{o} ; and if the field K' contains the field K as a subfield, then K' is said to be a *coefficient field over K* in \mathfrak{o} . It has been shown by Cohen²⁵ that while the ring \mathfrak{o} in general does not contain a coefficient field, the completion \mathfrak{o}^* of \mathfrak{o} always contains a coefficient field, provided $\mathfrak{o}/\mathfrak{m}$ has the same characteristic as \mathfrak{o} ; however, even in \mathfrak{o}^* there need not always exist a coefficient field over any given field K . We shall say that a generalized local ring \mathfrak{o} contains a *basic field*²⁶ K if \mathfrak{o}^* contains a coefficient field over K .

²³ Cohen [6].

²⁴ See Cohen [6], p. 61, Theorem 3.

²⁵ See Cohen [6], p. 72, Theorem 9.

²⁶ It is to be noticed that we use the expression "basic field" in a quite different sense from that of Chevalley in [2].

THEOREM 3. *Let U/K be a variety in S_m , and let (ξ) be a point in U/k such that $K(\xi)$ is a separably generated extension of K . Then the quotient ring $Q(U/K, (\xi))$ contains K as a basic field.*

Proof. Let r be the degree of transcendency of $K(\xi)$ over K . It is well known that we can always find a suitable affine transformation in S_m , such that after such a transformation the first r coordinates ξ_1, \dots, ξ_r of (ξ) are independent variables over K , and the field $K(\xi)$ is separably algebraic over $K(\xi_1, \dots, \xi_r)$. This implies that if $(x) = (x_1, \dots, x_m)$ is a generic point of U/K , then the x_1, \dots, x_r are independent variables over K and the quotient ring $Q(U/K, (\xi))$ contains the field $K(x_1, \dots, x_r)$. The residue ring of $Q(U/K, (\xi))$ over its maximal prime ideal, being isomorphic to the field $K(\xi)$, is then a separably algebraic extension of $K(x_1, \dots, x_r)$; and it is well known²⁷ that in this case the completion of $Q(U/K, (\xi))$ contains a coefficient field over $K(x_1, \dots, x_r)$, which is then evidently also a coefficient field over K .

Let \mathfrak{o} be a generalized local domain with a basic field K , \mathfrak{m} be the maximal prime ideal in \mathfrak{o} , and M be the quotient field of \mathfrak{o} . Let L be any field containing K , such that both L and M as well as L and $\mathfrak{o}/\mathfrak{m}$ are algebraically disjoint over K . According to Theorem 1, the Kronecker product $L \times \mathfrak{o}$ of L and \mathfrak{o} over K is an integral domain, and it contains \mathfrak{o} and L as two linearly disjoint subrings over K . It can be easily shown that $(L \times \mathfrak{o})\mathfrak{q} \cap \mathfrak{o} = \mathfrak{q}$ for every ideal \mathfrak{q} in \mathfrak{o} , and that $\bigcap_i (L \times \mathfrak{o})\mathfrak{q}_i = (L \times \mathfrak{o})(\bigcap_i \mathfrak{q}_i)$ for every set of ideals \mathfrak{q}_i in \mathfrak{o} . In particular, we have the relations $\bigcap_{s=1}^{\infty} (L \times \mathfrak{o})\mathfrak{m}^s = (L \times \mathfrak{o})(\bigcap_{s=1}^{\infty} \mathfrak{m}^s) = (0)$ and $(L \times \mathfrak{o})\mathfrak{m}^s \cap \mathfrak{o} = \mathfrak{m}^s$; which shows that if we consider $L \times \mathfrak{o}$ as a (generalized) $(L \times \mathfrak{o})\mathfrak{m}$ -adic ring in the sense of Zariski,²⁸ then $L \times \mathfrak{o}$ contains \mathfrak{o} as a subspace. Moreover, it is easily seen that the residue ring $L \times \mathfrak{o}/(L \times \mathfrak{o})\mathfrak{m}$ is isomorphic to the ring $L \times (\mathfrak{o}/\mathfrak{m})$ which is according to Theorem 1 an integral domain; this shows that $(L \times \mathfrak{o})\mathfrak{m}$ is a prime ideal in $L \times \mathfrak{o}$. Let \mathfrak{o}_L be the quotient ring of $L \times \mathfrak{o}$ with respect to the multiplicatively closed system $L \times \mathfrak{o} - (L \times \mathfrak{o})\mathfrak{m}$; then $\mathfrak{o}_L\mathfrak{m}$ is the ideal of non-units in \mathfrak{o}_L and has obviously a finite basis. We shall show now that \mathfrak{o}_L is a generalized local domain and contains \mathfrak{o} as a subspace; to do this it is sufficient to show that $\bigcap_{s=1}^{\infty} \mathfrak{o}_L\mathfrak{m}^s = (0)$ and $\mathfrak{o}_L\mathfrak{m}^s \cap \mathfrak{o} = \mathfrak{m}^s$.

²⁷ See Chevalley [2], p. 701, Proposition 3.

²⁸ See Zariski [9]; we add the word "generalized" to indicate the fact that the ring is not necessarily Noetherian.

In case K is a coefficient field of \mathfrak{o} , the residue ring $L \times \mathfrak{o}/(L \times \mathfrak{o})\mathfrak{m}$ is isomorphic to the field L ; this implies that the prime ideal $(L \times \mathfrak{o})\mathfrak{m}$ is maximal in $L \times \mathfrak{o}$ and hence the ideals $(L \times \mathfrak{o})\mathfrak{m}^s$ are all primary. It follows then that $\bigcap_{s=1}^{\infty} \mathfrak{o}_L \mathfrak{m}^s = \mathfrak{o}_L (\bigcap_{s=1}^{\infty} (L \times \mathfrak{o})\mathfrak{m}^s) = \mathfrak{o}_L (\bigcap_{s=1}^{\infty} \mathfrak{m}^s) = (0)$ and $\mathfrak{o}_L \mathfrak{m}^s \cap \mathfrak{o} = \mathfrak{o}_L \mathfrak{m}^s \cap L \times \mathfrak{o} \cap \mathfrak{o} = (L \times \mathfrak{o})\mathfrak{m}^s \cap \mathfrak{o} = \mathfrak{m}^s$. If K is not a coefficient field of \mathfrak{o} , then the completion \mathfrak{o}^* of \mathfrak{o} contains a coefficient[†] K' over K . Since K' is K -isomorphic to $\mathfrak{o}/\mathfrak{m}$, the fields L and K' are algebraically disjoint over K . Therefore the Kronecker product $L \times K'$ of L and K' over K is an integral domain; let L' be the field of quotients of $L \times K'$. Since K' is a coefficient field of \mathfrak{o}^* and L' contains K' , we can apply to L' and \mathfrak{o}^* over the field K' the case we have just proved and conclude that $\bigcap_{s=1}^{\infty} (\mathfrak{o}^*)_L \mathfrak{m}^s = (0)$ and $(\mathfrak{o}^*)_L \mathfrak{m}^s \cap \mathfrak{o}^* = \mathfrak{o}^* \mathfrak{m}^s$. Since the ring $(\mathfrak{o}^*)_L$ contains \mathfrak{o}_L as a subring, it follows then that $\bigcap_{s=1}^{\infty} \mathfrak{o}_L \mathfrak{m}^s \subset \mathfrak{o}_L \cap (\bigcap_{s=1}^{\infty} (\mathfrak{o}^*)_L \mathfrak{m}^s) = (0)$ and $\mathfrak{o}_L \mathfrak{m}^s \cap \mathfrak{o} \subset (\mathfrak{o}^*)_L \mathfrak{m}^s \cap \mathfrak{o} = \mathfrak{o}^* \mathfrak{m}^s \cap \mathfrak{o} = \mathfrak{m}^s$, and hence $\bigcap_{s=1}^{\infty} \mathfrak{o}_L \mathfrak{m}^s = (0)$ and $\mathfrak{o}_L \mathfrak{m}^s \cap \mathfrak{o} = \mathfrak{m}^s$.

Thus we have shown that in any case the ring \mathfrak{o}_L is a generalized local domain and contains the generalized local domain \mathfrak{o} as a subspace. We shall call \mathfrak{o}_L the *extension* of \mathfrak{o} over the field L . We remark further that in case L is a finitely generated extension of K (which is the only case of interest for our present purpose), it can be shown that \mathfrak{o}_L is a Noetherian ring if \mathfrak{o} is such; in other words, in this case \mathfrak{o}_L is a local domain if \mathfrak{o} is a local domain. However, this additional restriction is not necessary for our considerations here. It is easily seen that if a variety U/K has an extension over a field L containing K and if (ξ) is a point of U/K such that $K(\xi)$ and L are linearly disjoint over K , then the quotient ring $Q(U/L, (\xi))$ is the extension of the quotient ring $Q(U/K, (\xi))$ over the field L .

Consider the completion $(\mathfrak{o}_L)^*$ of the generalized local domain \mathfrak{o}_L ; $(\mathfrak{o}_L)^*$ is a local ring and contains the local ring \mathfrak{o}^* as a subring. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the minimal (unimbedded) prime divisors of the zero ideal in \mathfrak{o}^* . Then for each $i=1, \dots, n$, the residue ring $\mathfrak{o}^*/\mathfrak{p}_i$ is a complete local domain, and the residue ring $(\mathfrak{o}_L)^*/(\mathfrak{o}_L)^* \mathfrak{p}_i$ is a complete local ring and contains $\mathfrak{o}^*/\mathfrak{p}_i$ as a subring. We shall say that the field L is *analytically disjoint* with respect to the generalized local domain \mathfrak{o} over the field K , if the ideals $(\mathfrak{o}_L)^* \mathfrak{p}_i$, $i=1, \dots, n$, are all prime ideals in $(\mathfrak{o}_L)^*$, so that the rings $(\mathfrak{o}_L)^*/(\mathfrak{o}_L)^* \mathfrak{p}_i$, $i=1, \dots, n$, are all complete local domains.

Let \mathfrak{q} be any ideal in \mathfrak{o}^* ; we are concerned with the relations between

the complete local ring $(o_L)^*/(o_L)^*q$ and the complete local ring o^*/q . Let K' be a coefficient field over K in the complete local ring o^* , and let L' be the field of quotients of the Kronecker product $L \times K'$ over K . Then the Kronecker product $L' \times o^*$ of L' and o^* over K' is an m -adic ring, and the residue ring $L' \times o^*/(L' \times o^*)m$ is isomorphic with the subfield L' of $L' \times o^*$. It is then easily seen that every element in the set $L' \times o^* - (L' \times o^*)m$ is $\equiv 1 \pmod{(L' \times o^*)m}$, from which it follows²⁹ that the $(L' \times o^*)m$ -adic ring $L' \times o^*$ and the local ring $(o^*)_{L'}$ have the same completion. Since the completion of $(o^*)_{L'}$ is evidently equal to $(o_L)^*$, we have therefore $(o_L)^* = (L' \times o^*)^*$. If now q is any ideal ($\neq o^*$) in o^* , then the residue ring $L' \times o^*/(L' \times o^*)q$ contains L' and o^*/q as subrings and is $(L', o^*/q)$ -isomorphic with the Kronecker product $L' \times (o^*/q)$ over K' . Since the completion of $L' \times o^*/(L' \times o^*)q$ is the residue ring $(L' \times o^*)^*/(L' \times o^*)^*q = (o_L)^*/(o_L)^*q$, we conclude therefore that the two complete local rings $(o_L)^*/(o_L)^*q$ and $(L' \times (o^*/q))^*$ are $(L', o^*/q)$ -isomorphic with each other. In other words, the complete local ring $(o_L)^*/(o_L)^*q$ contains the Kronecker product $L' \times (o^*/q)$ of the two subrings L' and o^*/q over K' as a subspace and coincides with the adherence of the latter.

The following two theorems (in somewhat different form) are due to Chevalley:

THEOREM 4. *If o is a generalized local domain with a basic field K and if the residue field of o is a finitely generated extension of K , then any regular extension of K is analytically disjoint with respect to o over K .*

THEOREM 5. *If o is a generalized local domain with a coefficient field K , and if \bar{K} is analytically disjoint with respect to o over K , then any field containing K is analytically disjoint with respect to o over K .*

For proof we shall refer to Chevalley [2], § 4. It is sufficient to observe that if we set $(L' \times (o^*/p_i))^*$, o^*/p_i , L' , K' equal to \mathfrak{O} , o , Z , K in the notation of Chevalley [2], p. 77, then all the conditions there are fulfilled. If L is a regular extension of K (i. e. K is "strongly algebraically closed" in L , in the terminology of Chevalley), then L' is also a regular extension of K' , and Theorem 4 then follows from [2], p. 78, Proposition 9a and the remark immediately after it. Theorem 5 follows from [2], p. 80, Proposition 10b.

We shall also mention the following rather trivial theorem:

²⁹ See Zariski [9], p. 183. That our ring $L' \times o^*$ is not necessarily Noetherian is immaterial here.

THEOREM 6. *Let \mathfrak{o} be a generalized local domain containing a field K , and let L and L' be two fields such that $L \supset L' \supset K$. If L is analytically disjoint with respect to \mathfrak{o} over K , then \mathfrak{o}_L is defined and L is analytically disjoint with respect to \mathfrak{o}_L over L' .*

For, we have $(\mathfrak{o}_L)^* \supset (\mathfrak{o}_{L'})^* \supset \mathfrak{o}^*$ and $(\mathfrak{o}_L)^* \mathfrak{p}_i \cap (\mathfrak{o}_{L'})^* = (\mathfrak{o}_{L'})^* \mathfrak{p}_i$ for every i ; hence if the ideals $(\mathfrak{o}_L)^* \mathfrak{p}_i$ are prime in $(\mathfrak{o}_L)^*$, the ideals $(\mathfrak{o}_{L'})^* \mathfrak{p}_i$ must also be prime in $(\mathfrak{o}_{L'})^*$.

Let \mathfrak{o} be a generalized local domain with a coefficient field K , and let L be a field containing K such that the extension \mathfrak{o}_L of \mathfrak{o} over L is defined. The residue of $L \times \mathfrak{o} / (L \times \mathfrak{o})\mathfrak{m}$ is in this case isomorphic to L , and hence every element in the multiplicatively closed system $L \times \mathfrak{o} - (L \times \mathfrak{o})\mathfrak{m}$ is $\equiv 1 \pmod{(L \times \mathfrak{o})\mathfrak{m}}$. It follows then that the $(L \times \mathfrak{o})\mathfrak{m}$ -adic ring $L \times \mathfrak{o}$ and the local ring \mathfrak{o}_L have the same completion; that is, we have $(L \times \mathfrak{o})^* = (\mathfrak{o}_L)^*$ or also $(L \times \mathfrak{o}^*)^* = (\mathfrak{o}_L)^*$, since it is evident that $(L \times \mathfrak{o})^* = (L \times \mathfrak{o}^*)^*$. In case L is a finite algebraic extension of K , we can even write $(\mathfrak{o}_L)^* = L \times \mathfrak{o}^*$; for, in this case the \mathfrak{m} -adic ring $L \times \mathfrak{o}^*$ is already complete and hence coincides with $(L \times \mathfrak{o}^*)^*$.

THEOREM 7. *Let \mathfrak{o} be a generalized local domain with a coefficient field K , such that its completion \mathfrak{o}^* has no nilpotent elements. Let R be a ring containing \mathfrak{o}^* as a subring, and let L be a field in R such that L is a finite algebraic extension of K . If L is analytically disjoint with respect to \mathfrak{o} over K , then the subring $L\mathfrak{o}^*$ in R is (L, \mathfrak{o}^*) -isomorphic to $(\mathfrak{o}_L)^*$.*

Proof. It is clear that the subring $L\mathfrak{o}^*$ in R is an (L, \mathfrak{o}^*) -homomorphic image of $(\mathfrak{o}_L)^* = L \times \mathfrak{o}^*$. Therefore, in order to prove our theorem, it is only necessary to show that the subrings L and \mathfrak{o}^* in R are linearly disjoint over K . Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the prime divisors of the zero ideal in \mathfrak{o}^* ; the assumption that L is analytically disjoint with respect to \mathfrak{o} implies that each ring $L \times \mathfrak{o}^* / (L \times \mathfrak{o}^*)\mathfrak{p}_i$ is an integral domain. Since the ring $L \times \mathfrak{o}^* / (L \times \mathfrak{o}^*)\mathfrak{p}_i$ is $(L, \mathfrak{o}^* / \mathfrak{p}_i)$ -isomorphic to the ring $L \times (\mathfrak{o}^* / \mathfrak{p}_i)$, therefore $L \times (\mathfrak{o}^* / \mathfrak{p}_i)$ is also an integral domain; it follows then from Theorem 2 that any $(\mathfrak{o}^* / \mathfrak{p}_i)$ -homomorphism of $L \times (\mathfrak{o}^* / \mathfrak{p}_i)$ is an isomorphism. Since every element in \mathfrak{p}_i is annulled by every element in $\bigcap_{j \neq i} \mathfrak{p}_j$, it follows that every element

in $(L\mathfrak{o}^*)\mathfrak{p}_i \cap \mathfrak{o}^*$ is also annulled by every element in $\bigcap_{j \neq i} \mathfrak{p}_j$; and since

$(L\mathfrak{o}^*)\mathfrak{p}_i \cap \mathfrak{o}^* \supset \mathfrak{p}_i$ and since the ring \mathfrak{o}^* has no imbedded prime divisors of the zero ideal, it follows that $(L\mathfrak{o}^*)\mathfrak{p}_i \cap \mathfrak{o}^* = \mathfrak{p}_i$. It follows then that the residue ring $L\mathfrak{o}^* / (L\mathfrak{o}^*)\mathfrak{p}_i$ contains L and $\mathfrak{o}^* / \mathfrak{p}_i$ as subrings, and it is an $(L, \mathfrak{o}^* / \mathfrak{p}_i)$ -homomorphic image of $L \times (\mathfrak{o}^* / \mathfrak{p}_i)$ and hence also an $(L, \mathfrak{o}^* / \mathfrak{p}_i)$ -

isomorphic image of $L \times (\mathfrak{o}^*/\mathfrak{p}_i)$. Thus each $L\mathfrak{o}^*/(L\mathfrak{o}^*)\mathfrak{p}_i$ is a complete local domain and contains L and $\mathfrak{o}^*/\mathfrak{p}_i$ as two linearly disjoint subrings over K .

Let u_1, \dots, u_m be a set of linearly independent elements over K in \mathfrak{o}^* , and denote by (u_1, \dots, u_m) the linear space determined by this set over K . If l_1 is the dimension of the sub-space of (u_1, \dots, u_m) which is contained in the ideal \mathfrak{p}_1 , then we can replace the set u_1, \dots, u_m by a linearly equivalent set over K , such that the space (u_1, \dots, u_{l_1}) is in \mathfrak{p}_1 and the space (u_{l_1+1}, \dots, u_m) contains no elements in \mathfrak{p}_1 . If l_2 is the dimension of the subspace of (u_1, \dots, u_{l_1}) which is contained in \mathfrak{p}_2 , then we can replace the set u_1, \dots, u_{l_1} by a linearly equivalent set over K , such that the space (u_1, \dots, u_{l_2}) is in $\mathfrak{p}_1 \cap \mathfrak{p}_2$ and the space $(u_{l_2+1}, \dots, u_{l_1})$ contains no elements in \mathfrak{p}_2 . Thus we can proceed until in the final stage we have a set $u_1, \dots, u_{l_{n-1}}$ such that $(u_1, \dots, u_{l_{n-1}})$ is contained in $\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_{n-1}$; here the process must stop and the space $(u_1, \dots, u_{l_{n-1}})$ contains no elements in \mathfrak{p}_n , for we have $\bigcap_{i=1}^n \mathfrak{p}_i = (0)$ on account of the absence of nilpotent elements in \mathfrak{o}^* . Adding for the sake of convenience an element $u_0 = 0$, our final results will be a set u_0, \dots, u_m , which is linearly equivalent over K to the original set, but has the property that for each $i = 0, \dots, n$, the space (u_0, \dots, u_{l_i}) is contained in $\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_i$ and the space $(u_{l_i+1}, \dots, u_{l_{i-1}})$ contains no elements in \mathfrak{p}_i (where we set $l_0 = m$ and $l_n = 0$). We shall now show that if there is a linear relation $\sum_{j=1}^m c_j u_j = 0$ with coefficients c_j in L , then all the c_j must vanish. Assume that it has already been proved that $c_j = 0$ for all $j > l_{i-1}$, we shall show that $c_j = 0$ for all $j > l_i$. In fact, if not all $c_{l_{i+1}}, \dots, c_{l_{i-1}}$ are zero, then the relation $\sum_{j=1}^{l_{i-1}} c_j u_j = 0$ will induce a linear relation $\sum_{j=l_{i+1}}^{l_{i-1}} c_j u_j = 0$ in $L\mathfrak{o}^*/(L\mathfrak{o}^*)\mathfrak{p}_i$ between the $u_{l_{i+1}}, \dots, u_{l_{i-1}}$, considered as elements in $\mathfrak{o}^*/\mathfrak{p}_i$. Since in $L\mathfrak{o}^*/(L\mathfrak{o}^*)\mathfrak{p}_i$ the subrings L and $\mathfrak{o}^*/\mathfrak{p}_i$ are linearly disjoint over K , it follows that the $u_{l_{i+1}}, \dots, u_{l_{i-1}}$ must be linearly dependent over K in $\mathfrak{o}^*/\mathfrak{p}_i$; this means that there is a linear combination of the $u_{l_{i+1}}, \dots, u_{l_{i-1}}$ with coefficients in K which is an element in \mathfrak{p}_i , in contradiction to the fact that the space $(u_{l_{i+1}}, \dots, u_{l_{i-1}})$ contains no elements in \mathfrak{p}_i . Thus we have shown by induction that all the c_1, \dots, c_m must vanish, and hence the elements u_1, \dots, u_m are linearly independent over L . This concludes the proof of our theorem.

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ARITHMETICAL PROPERTIES OF THE ELLIPTIC POLYNOMIALS ARISING FROM THE REAL MULTIPLICATION OF THE JACOBI FUNCTIONS.*

By MORGAN WARD.

I. Introduction.

1. In a previous paper in this JOURNAL,¹ referred to hereafter as "M," I have made a detailed investigation of the arithmetical properties of the sequence of polynomials (ψ) ,

$$\psi_n = \psi_n(\wp(u); g_2, g_3), \quad (n = 0, 1, 2, \dots)$$

associated with the real multiplication of the Weierstrass \wp function when $\wp(u)$, g_2 and g_3 are given fixed rational values. If the elliptic discriminant $g_2^3 - 27g_3^2$ vanishes, (ψ) reduces essentially to Lucas' well-known linear sequence (U) ,

$$U_n = (\alpha^n - \beta^n)/(\alpha - \beta), \quad (n = 0, 1, 2, \dots).$$

I study here the arithmetical properties of the four polynomials A_n , B_n , C_n , and D_n associated with the real multiplication of Jacobi's sn , cn , and dn .²

Here each of A_n, \dots, D_n is a polynomial in sn^2u and k^2 with rational integral coefficients. Consequently, if we substitute for sn^2u and k^2 two fixed algebraic numbers, we obtain four sequences of algebraic numbers (A) , (B) , (C) and (D) . The arithmetical properties of these elliptic sequences are the subject of this investigation. If k^2 is zero or one, the four sequences reduce essentially to Lucas' sequences (U) and (V) , where $V_n = \alpha^n + \beta^n$.

2. To give an idea of the type of results obtained, choose fixed rational integral values x_0 and a_0 for sn^2u and k^2 . Then the four elliptic sequences consist exclusively of rational integers. Each sequence is numerically periodic modulo m for any modulus m , but only the sequence (B) is an elliptic

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¹ See the reference Ward [1] at the close of this paper.

² If n is an odd integer

$$snnu/snu = B_n/A_n, \quad cnnu/cnu = C_n/A_n, \quad dnu = D_n/A_n$$

with similar formulas when n is even.

divisibility sequence in the sense of M. The only primes whose laws of apparition present new features of interest are those dividing neither $2a_0(1 - a_0)$ nor $x_0(1 - x_0)(1 - a_0x_0)$. Let p be such a prime, and let its rank of apparition in the divisibility sequence (B) be ρ , so that $B_n \equiv 0 \pmod{p}$ if and only if $n \equiv 0 \pmod{\rho}$. Then if ρ is odd, p divides no term of (A) , (C) or (D) . But if ρ is even, it appears as a divisor of precisely one of the sequences (A) , (C) and (D) . Suppose for example that it is a divisor of (C) . Then no term of (A) or (D) is divisible by p , and $C_n \equiv 0 \pmod{p}$ if and only if n is an odd multiple of $\rho/2$.

The laws of repetition and apparition for powers of primes in (A) , (B) , (C) and (D) are easily reduced to the corresponding laws for (B) which in turn are corollaries of the results in Ward [2] for elliptic divisibility sequences.

3. The plan of the paper is sufficiently clear from the chapter titles. Accounts of the elliptic polynomials A_n, \dots, D_n , are given in Krause [1] and Fricke [2]; but there are errors in the formulas given in these works. Although the properties of the Al functions³ on which we base the theory were very completely worked out in Weierstrass [1], most of the formulas which we utilize are most simply obtained by transformation from the corresponding σ or θ function formulas.

II. Properties of Weierstrass Al Functions.

4. The multiplication theory of the Jacobian elliptic functions is most conveniently developed in terms of certain modified θ functions, the Al functions of Weierstrass (Weierstrass, [1]). These may be defined as follows: Let v be a complex variable, $q = e^{\pi i \tau}$ with $\text{Im } \tau > 0$, $u = 2Kv$, where K is the complete elliptic integral. The Jacobi theta functions are then

$$\begin{aligned}\Theta(u) &= \theta_0(v) = \sum_{-a}^{+a} (-1)^m q^{m^2} e^{2m\pi i v}, \\ H(u) &= \theta_1(v) = -i \sum_{-a}^{+a} (-1)^m q^{(m+\frac{1}{2})^2} e^{(2m+1)\pi i v}, \\ H_1(u) &= \theta_2(v) = \sum_{-a}^{+a} q^{(m+\frac{1}{2})^2} e^{(2m+1)\pi i v}, \\ \Theta_1(u) &= \theta_3(v) = \sum_{-a}^{+a} q^{m^2} e^{2m\pi i v}.\end{aligned}$$

³ The functions were named by Weierstrass in honor of Abel who was the first to consider them.

If we write θ_α for $\theta_\alpha(0)$ and θ_0 for $\theta_0(0)$, then Weierstrass Al functions may be defined by

$$(4.1) \quad \begin{aligned} Al_1(u) &= \theta_3/\theta_0\theta_2 \exp(-\tfrac{1}{2}v^2\theta''_0/\theta_0)\theta_1(v), \\ Al_\alpha(u) &= 1/\theta_\alpha \exp(-\tfrac{1}{2}v^2\theta''_0/\theta_0)\theta_\alpha(v), \end{aligned} \quad (\alpha = 0, 2, 3).$$

Note that $Al_1(u)$ is odd and $Al_2(u)$, $Al_3(u)$ and $Al_0(u)$ even. Also

$$(4.11) \quad Al_1(0) = 0, \quad Al_\alpha(0) = 1, \quad (\alpha = 0, 2, 3).$$

sn , cn and dn have particularly simple expressions in terms of the Al s; namely

$$(4.2) \quad snu = Al_1(u)/Al_0(u), \quad cnu = Al_2(u)/Al_0(u), \quad dnu = Al_3(u)/Al_0(u).$$

The relationship to the Weierstrass σ functions is also very simple; namely if $w = \omega u/K$, then

$$(4.3) \quad \begin{aligned} Al_1(u) &= (e_1 - e_3)^{\frac{1}{2}} e^{e_3 w^2/2} \sigma(w); & Al_3(u) &= e^{e_3 w^2/2} \sigma_2(w); \\ Al_2(u) &= e^{e_3 w^2/2} \sigma_1(w); & Al_0(u) &= e^{e_3 w^2/2} \sigma_3(w). \end{aligned}$$

For the lemniscate case, $e_3 = 0$ and the Al functions are essentially the σ functions.

The fundamental three-terms sigma identity becomes

$$(4.4) \quad \begin{aligned} Al_1(u + u_1) Al_1(u - u_1) Al_1(u_2 + u_3) Al_1(u_2 - u_3) \\ + Al_1(u + u_2) Al_1(u - u_2) Al_1(u_3 + u_1) Al_1(u_3 - u_1) \\ + Al_1(u + u_3) Al_1(u - u_3) Al_1(u_1 + u_2) Al_1(u_1 - u_2) = 0. \end{aligned}$$

5. We next introduce four new functions $K_{\alpha n}(u)$ of u and n by the definition

$$(5.1) \quad K_{\alpha n} = Al_\alpha(nu)/Al_0(u)^{n^2}, \quad (n = 0, 1, 2, \dots; \alpha = 0, 1, 2, 3).$$

Evidently

$$(5.2) \quad snnu = K_{1n}/K_{0n}, \quad cnnu = K_{2n}/K_{0n}, \quad dnnu = K_{3n}/K_{0n}.$$

The first three initial values of the four sequences (K_α) are as follows:

TABLE I. Initial values of $K_{\alpha n}$.

α/n	0	1	2
0	1	1	$1 - k^2 sn^4 u.$
1	0	snu	$2snu \ cnu \ dnu.$
2	1	cnu	$1 - 2sn^2 u + k^2 sn^4 u.$
3	1	dnu	$1 - 2k^2 sn^2 n + k^2 sn^4 u.$

There are twenty-four addition formulas for the products $Al_a(u+v) \times Al_\beta(u-v)$ obtainable by simple transformations from the corresponding formulas for $\theta_a(u+v)\theta_\beta(u-v)$ or by suitably specializing (4.4). If in these formulas we replace u by nu and v by mu and divide by $Al_0(u)^{2(n^2+m^2)}$, we obtain on using (5.1) twenty-four addition formulas for the products $K_{an+m}K_{\beta n-m}$. It is sufficient to quote here a few such formulas as examples:

 TABLE II. Addition formulas for K_{an} .

- (i) $K_{0n+m}K_{0n-m} = K_{0n}^2 K_{0m}^2 - k^2 K_{1n}^2 K_{1m}^2$

 (v) $K_{1n+m}K_{1n-m} = K_{1n}^2 K_{0m}^2 - K_{0n}^2 K_{1m}^2$

 (xix) $K_{0n+m}K_{1n-m} = K_{0n}K_{1n}K_{2m}K_{3m} - K_{2n}K_{3n}K_{0m}K_{1m}$

 (xxiv) $K_{2n+m}K_{3n-m} = K_{2n}K_{3n}K_{2m}K_{3m} - k'^2 K_{0n}K_{1n}K_{0m}K_{1m}$

In formula (xxiv), k'^2 is the complementary modulus $1 - k^2$.

If we take $m = \pm n$ or $m = n$ and $n = n + 1$ in the addition formulas, we obtain a set of over forty "duplication formulas" which it is also unnecessary to give in detail; the two formulas so obtained from (i) and (xix) suffice as examples:

$$(5.3) \quad K_{12n+1}K_{11} = K_{1n+1}^2 K_{0n}^2 - K_{0n+1}^2 K_{1n}^2,$$

$$(5.4) \quad K_{12n} = 2K_{0n}K_{1n}K_{2n}K_{3n}.$$

As noted in Krause [1], the duplication formulas allow many results about the algebraic form of the polynomials K_{an} to be proved by mathematical induction from the initial values given in Table I; the results in the next section are easily obtained in this manner.

III. The Four Elliptic Polynomials.

6. If we write

$$(6.1) \quad \begin{aligned} K_{0n}(u) &= A_n(sn^2u; k^2) & n \text{ odd or even;} \\ &snub_n(sn^2u; k^2) & n \text{ odd;} \\ K_{1n}(u) &= & \\ &snuc_n dnuB_n(sn^2u; k^2) & n \text{ even;} \\ &cnuC_n(sn^2u; k^2) & n \text{ odd;} \end{aligned}$$

$$\begin{aligned}
 K_{2n}(u) &= & C_n(sn^2u; k^2) & n \text{ even}; \\
 & & dnuD_n(sn^2u; k^2) & n \text{ odd}; \\
 K_{3n}(u) &= & D_n(sn^2u; k^2) & n \text{ even};
 \end{aligned}$$

then (Krause [1], pp. 159-162, Fricke [2], Chapter 2) A_n , B_n , C_n and D_n are polynomials in sn^2u and k^2 with rational integral coefficients. It is convenient to let

$$(6.2) \quad x = sn^2u, \quad a = k^2.$$

Then

$$(6.3) \quad K_{1n}(u) = \begin{aligned} & (x)^{\frac{1}{2}} B_n(x; a), & n \text{ odd}; \\ & (x(1-x)(1-ax))^{\frac{1}{2}} B_n(x; a), & n \text{ even}; \end{aligned}$$

with similar formulas for K_{0n} , K_{2n} and K_{3n} .

We shall refer to A_n , B_n , C_n and D_n as the "elliptic polynomials of order n ." If we let

$$(6.4) \quad \alpha_n = \begin{aligned} & (n^2 - 1)/2, & n \text{ odd}, \\ & n^2/2, & n \text{ even}, \end{aligned} \quad \beta_n = \begin{aligned} & (n^2 - 1)/2 & n \text{ odd}, \\ & (n^2 - 4)/2 & n \text{ even}, \end{aligned}$$

then A_n , C_n and D_n are of degree α_n in x and B_n is of degree β_n in x .

7. There are a number of transformation formulas for the elliptic polynomials (Krause [1], Chapter III, Fricke [1], [2]) which are of arithmetical importance. These arise either by increasing u in the Al and K_n functions by the quarter periods K , iK' and $K + iK'$ or by performing the fundamental substitutions $\tau \rightarrow \tau + 1$, $\tau \rightarrow -1/\tau$ of the modular group on the four Al functions. It suffices here to develop one formula of each type by way of example.

Weierstrass showed that

$$\begin{aligned}
 Al_0(u + K) &= 1/k'^{\frac{1}{2}} e^{\lambda(u^2 - (u+K)^2)} Al_3(u), \\
 Al_3(u + 2K) &= e^{\lambda(u^2 - (u+K)^2)} Al_3(u),
 \end{aligned}$$

where we have written λ for $(K - E)/2K$.

It easily follows that if n is odd,

$$Al_0(u + nK) = 1/k'^{\frac{1}{2}} e^{\lambda(u^2 - (u+nK)^2)} Al_3(u).$$

Hence

$$\begin{aligned} K_{0n}(u+K) \\ = Al_0(nu+nK)/Al_0(u+K)^{n^2} = k'^{(n^2-1)/2}(Al_3(nu)/Al_3(u))^{n^2}. \end{aligned}$$

On multiplying both sides of this expression by $dnu^{n^2} = (Al_3(u)/Al_0(u))^{n^2}$, we find that

$$(7.1) \quad dnu^{n^2}K_{0n}(u+K) = k'^{(n^2-1)/2}K_{3n}(u).$$

Now $sn(u+K) = cnu/dnu$. Hence the substitution of $u+K$ for u induces the substitution of $(1-x)/(1-ax)$ for $x = sn^2u$. Therefore we obtain from (7.2) on substituting for K_{an} their expressions in terms of A_n and D_n the transformation formula

$$(1-ax)^{an}A_n((1-x)/(1-ax)) = (1-a)^{an/2}D_n(x) \quad n \text{ odd.}$$

The following sets of transformation formulas are obtained by proceeding systematically in this manner.

8. The modular transformation $\tau \rightarrow -1/\tau$ is simply Jacobi's imaginary transformation $u \rightarrow iu$, $k \rightarrow k'$. Now Weierstrass⁴ showed that

$$Al_3(iu; k') = e^{u^2/2}Al_3(u; k)$$

$$Al_0(iu; k') = e^{u^2/2}Al_2(u; k)$$

Hence

$$K_{3n}(iu; k') = Al_3(niu; k')/Al_0(iu; k')^{n^2} = Al_3(nu; k)/Al_2(u; k)^{n^2},$$

or

$$(8.1) \quad K_{3n}(iu; k') = cnu^{-n^2}K_{3n}(u; k).$$

But since $sn(iu, k') = isn(u, k)/cn(u, k)$, Jacobi's imaginary transformation induces the transformation

$$(8.2) \quad x \rightarrow -x/(1-x), \quad a \rightarrow 1-a$$

on x and a .

Now $K_{3n} = (1-ax)^{\frac{1}{2}}D_n$ or D_n according as n is odd or even, and (8.2) throws $\sqrt{1-ax}$ into $(1-ax)^{\frac{1}{2}}/1-x$. Hence on substituting into (8.1) and using the abbreviation α_n for $(n^2-1)/2$ or $n^2/2$ we obtain the formula

$$(1-x)^{\alpha_n}D_n(x/(x-1); 1-a) = D_n(x; a).$$

The formulas listed below were obtained by systematically combining

⁴ Weierstrass [1], page 20.

TABLE III. Transformation formulas.

	<i>n</i> odd	<i>n</i> even
$(1-ax)^{\alpha_n} A_n((1-x)/(1-ax)) = (1-a)^{\alpha_n/2} D_n(x);$		$= (1-a)^{\alpha_n/2} A_n(x)$
$(1-ax)^{\beta_n} B_n((1-x)/(1-ax)) = (-1)^{(n-1)/2} (1-a)^{\beta_n/2} C_n(x);$		$= (-1)^{(n-2)/2} (1-a)^{\beta_n/2} (x)$
$(1-ax)^{\alpha_n} C_n((1-x)/(1-ax)) = (-1)^{(n-1)/2} (1-a)^{\alpha_n/2} B_n(x);$		$= (-1)^{n/2} (1-a)^{\alpha_n/2} C_n(x)$
$(1-ax)^{\alpha_n} D_n((1-x)/(1-ax)) = (1-a)^{\alpha_n/2} A_n(x);$		$= (1-a)^{\alpha_n/2} D_n(x)$
$(\sqrt{ax})^{\alpha_n} A_n(1/ax) = (-1)^{(n-1)/2} B_n(x);$		$= (-1)^{n/2} A_n(x)$
$(\sqrt{ax})^{\beta_n} B_n(1/ax) = (-1)^{(n-1)/2} A_n(x);$		$= (-1)^{(n-2)/2} B_n(x)$
$(\sqrt{ax})^{\alpha_n} C_n(1/ax) = D_n(x);$		$= C_n(x)$
$(\sqrt{ax})^{\alpha_n} D_n(1/ax) = C_n(x);$		$= D_n(x)$
$(1-x)^{\alpha_n} A_n((1-ax)/(a-ax)) = ((1-a)/a)^{\alpha_n/2} C_n(x);$		$= (-1)^{n/2} ((1-a)/a)^{\alpha_n/2} A_n(x)$
$(1-x)^{\beta_n} B_n((1-ax)/(a-ax)) = (-1)^{(n-1)/2} ((1-a)/a)^{\beta_n/2} D_n(x);$		$= ((1-a)/a)^{\beta_n/2} B_n(x)$
$(1-x)^{\alpha_n} C_n((1-ax)/(a-ax)) = ((1-a)/a)^{\alpha_n/2} A_n(x);$		$= (-1)^{n/2} ((1-a)/a)^{\alpha_n/2} C_n(x)$
$(1-x)^{\alpha_n} D_n((1-ax)/(a-ax)) = (-1)^{(n-1)/2} ((1-a)/a)^{\alpha_n/2} B_n(x);$		$= ((1-a)/a)^{\alpha_n/2} D_n(x).$

the two transformations $\tau \rightarrow -1/\tau$ and $\tau \rightarrow 1 + \tau$; the latter transformation induces the transformation

$$(8.3) \quad x \rightarrow (x - ax)/(1 - ax), \quad a \rightarrow a/(a - 1)$$

on x and a .

TABLE IV. Transformation formulas.

$$\begin{aligned} A_n(x; a) &= (1 - ax)^{an} D_n((x - ax)/(1 - ax); a/(a - 1)) \\ &= (1 - x)^{an} C_n(x/(x - 1); 1 - a), \\ &= (1 - ax)^{an} C_n(ax/(ax - 1); (a - 1)/a) = A_n(ax; 1/a) \\ &= (1 - x)^{an} D_n((ax - x)/(1 - x); 1/(1 - a)). \end{aligned}$$

$$\begin{aligned} B_n(x; a) &= (1 - ax)^{\beta n} B_n((x - ax)/(1 - ax); a/(a - 1)) \\ &= (1 - x)^{\beta n} B_n(x/(x - 1); 1 - a), \\ &= (1 - ax)^{\beta n} B_n(ax/(ax - 1); (a - 1)/a) = B_n(ax; 1/a) \\ &= (1 - x)^{\beta n} B_n((ax - x)/(1 - x); 1/(1 - a)). \end{aligned}$$

$$\begin{aligned} (8.4) \quad C_n(x; a) &= (1 - ax)^{an} C_n((x - ax)/(1 - ax); a/(a - 1)) \\ &= (1 - x)^{an} A_n(x/(x - 1); 1 - a), \\ &= (1 - ax)^{an} D_n(ax/(ax - 1); (a - 1)/a) = D_n(ax; 1/a) \\ &= (1 - x)^{an} A_n((ax - x)/(1 - x); 1/(1 - a)). \end{aligned}$$

$$\begin{aligned} D_n(x; a) &= (1 - ax)^{an} A_n((x - ax)/(1 - ax); a/(a - 1)) \\ &= (1 - x)^{an} D_n(x/(x - 1); 1 - a), \\ &= (1 - ax)^{an} A_n(ax/(ax - 1); (a - 1)/a) = C_n(ax; 1/a) \\ &= (1 - x)^{an} C_n((ax - x)/(1 - x); 1/(1 - a)). \end{aligned}$$

We finally tabulate for later reference the first few initial values of the four elliptic sequences.

TABLE V. Initial values.

n	0	1	2	3
A_n	1	1	$1 - ax^2$	$1 - 6ax^2 + 4a(1 + a)x^3 - 3a^2x^4$
B_n	0	1	2	$3 - 4(1 + a)x + 6ax^2 - a^2x^4$
C_n	1	1	$1 - 2x + ax^2$	$1 - 4x + 6ax^2 - 4a^2x^3 + a^2x^4$
D_n	1	1	$1 - 2ax + ax^2$	$1 - 4ax + 6ax^2 - 4ax^3 + a^2x^4$

The transformation formulas of Tables III and IV may all be proved without function theory by mathematical induction from the duplication formulas and the initial values in Table V.

IV. Elliptic Divisibility Sequences in Domains of Integrality.

9. In M , the functional equation

$$(9.1) \quad \omega_{m+n}\omega_{m-n} = \omega_{m+1}\omega_{m-1}\omega_n^2 - \omega_{n+1}\omega_{n-1}\omega_m^2$$

was solved completely over the ring of rational integers and over the field of all complex numbers. It is necessary for the purposes of this paper to extend certain theorems of M to solutions of (9.1) over more general rings.

Let \mathcal{R} denote a domain of integrality; (commutative ring with a unit and no divisors of zero). \mathcal{R} may be a field; in any event we denote its quotient field by \mathcal{F} , and for brevity refer to \mathcal{R} as a ring. We are interested in solutions of (9.1) over \mathcal{R} and over \mathcal{F} . We lay down the following definitions:

A particular solution

$$(h): h_0, h_1, h_2, \dots, h_n, \dots$$

of (9.1) will be said to belong to \mathcal{R} (to \mathcal{F}) if all its terms belong to \mathcal{R} (to \mathcal{F}). If a and b belong to \mathcal{R} , we say that a divides b in \mathcal{R} if there exists an element c of \mathcal{R} such that $ac = b$. We write: $a \mid b$ (\mathcal{R}).

In particular, if \mathcal{R} is a field, and $b \neq 0$, $a \mid b$ (\mathcal{R}) for every $a \neq 0$.

If m is an ideal of \mathcal{R} , $a \equiv b$ (m) means as usual $a - b$ is contained in m .

If p is a maximal ideal of \mathcal{R} , the quotient ring \mathcal{R}/p is a field. If this field is finite its order is a power of a certain rational prime p . We denote the order by $Np = p^f$ and call p the (rational) prime belonging to p .

Definition 9.1. A solution of (9.1) is said to be "regular over \mathcal{F} " if it belongs to \mathcal{F} and if

$$(9.2) \quad h_0 = 0, \quad h_1 = 1, \quad h_2, h_3 \text{ not both zero.}$$

Definition 9.2. A solution (h) of (9.1) is said to be "a divisibility sequence over \mathcal{R} " if it belongs to \mathcal{R} and if

$$(9.3) \quad h_r \mid h_s (\mathcal{R}) \quad \text{whenever } r \mid s.$$

Let (h) belong to \mathcal{R} , and let m be an ideal of \mathcal{R} . m is called a divisor of (h) if it contains at least one term h_{n_0} of the sequence (h) with $n_0 > 0$. n_0 is called a "place of apparition" of m in (h) . If in addition $h_r \not\equiv 0 \pmod{m}$ for every proper divisor r of n_0 , then n_0 is called a rank of apparition of m in (h) .

10. The proofs of the theorems which follow are by mathematical

induction with the exception of the proof of Theorem 10.7 which uses the Dirichlet box principle. In any event they are almost word for word the same as corresponding theorems in M for the special cases when \mathcal{R} is the ring of rationals or the complex field. We shall accordingly cite the corresponding results in M for the details of proof.

THEOREM 10.1. *Let (h) be a sequence satisfying the following three conditions:*

$$(10.1) \quad (h) \text{ is a regular solution of (9.1) over } \mathcal{F},$$

$$(10.2) \quad h_2, h_3 \text{ and } h_4 \text{ belong to } \mathcal{R},$$

$$(10.3) \quad h_2 \mid h_4 (\mathcal{R}).$$

Then (h) is a divisibility sequence over \mathcal{R} uniquely determined by its initial values h_2, h_3 and h_4 .

Proof. M , Theorem 4.1. Chapter II.

THEOREM 10.2. *Under the hypotheses of Theorem 10.1,*

$$(10.4) \quad h_n = P_n(h_2, h_3, h_4),$$

where P_n is a polynomial in h_2, h_3, h_4 with rational integral coefficients and such that for any element a of \mathcal{F}

$$(10.5) \quad P_n(a^3 h_2, a^8 h_3, a^{15} h_4) = a^{n^2-1} P_n(h_2, h_3, h_4).$$

Proof. M , Theorem 4.1. Chapter II.

THEOREM 10.3. *If (k) is any regular solution of (9.1) over \mathcal{F} and a any non-zero element of \mathcal{F} , then (l) is also a regular solution over \mathcal{F} , where*

$$l_n = a^{n^2-1} k_n, \quad (n = 0, 1, 2, \dots).$$

THEOREM 10.4. *If (k) is any regular solution of (1.1) over \mathcal{F} , then there always exists an element a of \mathcal{F} and a regular solution (h) of (9.1) over \mathcal{R} satisfying conditions (10.1), (10.2) and (10.3) such that*

$$k_n = a^{n^2-1} h_n, \quad (n = 0, 1, 2, \dots).$$

Proof. M , Theorems 21.1, 21.3.

THEOREM 10.5. *Let (k) be a regular solution of (1.1) over \mathcal{F} with $k_2 \neq 0$. Then if two consecutive terms of (k) are zero, all terms vanish beyond the third.*

THEOREM 10.6. *Let (h) be a solution of (1.1) over \mathcal{R} with $h_0 = 0$ and $h_1 = 1$ (but not necessarily a regular solution). Then if (h) is a divisibility sequence over \mathcal{R} and two consecutive terms of (h) vanish, all terms of (h) vanish beyond the third.*

Proof. M, Lemma 4.1.

It is shown in M that the hypothesis that (h) is a divisibility sequence in Theorem 10.6 is necessary for the truth of the theorem when (h) is not regular; that is, when both h_2 and h_3 are zero.

THEOREM 10.7. *Let (h) be an elliptic divisibility sequence over \mathcal{R} , and let \mathfrak{p} be a prime ideal of \mathcal{R} whose quotient ring \mathcal{R}/\mathfrak{p} is of finite order $N\mathfrak{p}$. Then \mathfrak{p} is a divisor of (h) , and has a rank of apparition r in (h) less than $2(N\mathfrak{p} + 1)$.*

Proof. M, Theorem 5.1.

It is shown in M that the upper limit $2(N\mathfrak{p} + 1)$ is the best possible for the rank of \mathfrak{p} in (h) .

V. The Laws of Apparition of Prime Ideals in Elliptic Sequences.

11. The connection between the results of Chapter IV and the elliptic polynomials is made by the following theorem.

THEOREM 11.1. *If u_0 is neither a zero nor a pole of snu and if*

$$(11.1) \quad h_n = K_{1n}(u_0)/K_{11}(u_0) = K_{1n}/K_{11} \quad (n = 0, 1, 2, \dots)$$

then the sequence (h) is a solution of the functional equation

$$(9.1) \quad \omega_{m+n}\omega_{m-n} = \omega_{m+1}\omega_{m-1}\omega_n^2 - \omega_{n+1}\omega_{n-1}\omega_m^2$$

over the complex field.

Proof. Let l, m and n be fixed integers. Take $u = 0, u_1 = lu_0, u_2 = mu_0$ and $u_3 = nu_0$ in the basic three-term identity (4.4), and divide by the non-zero quantity $Al_0(u_0)^{2(l^2+m^2+n^2)}$. Then we obtain on substitution, from (5.1) the formula

$$K_{21l}K_{1m+n}K_{1m-n} + K_{21m}K_{1n+l}K_{1n-l} + K_{21n}K_{1l+m}K_{1l-m} = 0.$$

Now $K_{11} = snu_0 \neq 0$. Hence on letting $l = 1$ and dividing by K_{11}^4 , we obtain from (11.1) the formula

$$h_{m+n}h_{m-n} = h_{m+1}h_{m-1}h_n^2 - h_{n+1}h_{n-1}h_m^2;$$

for $K_{1l-m} = -K_{1m-l}$.

We shall now assign to x and to a algebraic integer values x_0 and a_0 . Let \mathcal{F}_1 be the field $F[x_0, a_0]$ obtained by adjoining x_0 and a_0 to the rational field F , and let \mathcal{R}_1 be the ring of integers of \mathcal{F}_1 . Let \mathcal{F} be the field $F[x_0^{\frac{1}{2}}, 1-x_0^{\frac{1}{2}}, 1-a_0x_0^{\frac{1}{2}}]$, and \mathcal{R} the ring of integers of \mathcal{F} . Clearly $\mathcal{R} \supseteq \mathcal{R}_1$, $\mathcal{F} \supseteq \mathcal{F}_1$, and every ideal of either ring has a finite norm. We shall use German letters to denote either ideals of \mathcal{R} or of \mathcal{R}_1 , and let \mathfrak{p} denote as usual a prime ideal, and p the corresponding rational prime.

The four elliptic sequences (A), (B), (C) and (D) belong to \mathcal{R}_1 while the four sequences (K_α), ($\alpha = 0, 1, 2, 3$) belong to \mathcal{R} .

Let u_0 be chosen in a period parallelogram so that ⁵

$$\operatorname{sn} u_0 = (x_0)^{\frac{1}{2}}.$$

Then

$$(11.2) \quad h_n = \begin{cases} B_n(x_0, a_0), & n \text{ odd,} \\ ((1-x_0)(1-a_0x_0))^{\frac{1}{2}} B_n(x_0, a_0), & n \text{ even.} \end{cases}$$

Hence the initial values of (h) are

$$\begin{aligned} h_0 &= 0, & h_1 &= 1, & h_2 &= 2((1-x_0)(1-a_0x_0))^{\frac{1}{2}}, \\ h_3 &= B_3(x_0, a_0), & h_4 &= ((1-x_0)(1-a_0x_0))^{\frac{1}{2}} B_4(x_0, a_0). \end{aligned}$$

Now by direct calculation or from Table III,

$$(11.3) \quad B_3(1; a_0) = -(1-a_0)^2, \quad B_3(1/a_0; a_0) = -(1-a_0/a_0^2)^2.$$

Hence h_2 and h_3 both vanish if and only if $x_0 = 1$ and $a_0 = 1$. Also $h_4 = 2h_2A_2C_2D_2$ by formula (5.4). Hence $h_2 \mid h_4 (\mathcal{R})$.

Thus the sequence (h) satisfies all the hypotheses of Theorem 10.1. We may consequently state

THEOREM 11.1. *Unless both x_0 and a_0 are unity, the sequence (h) defined by*

$$(11.2) \quad h_n = B_n(x_0; a_0), \quad n \text{ odd;} \\ = ((1-x_0)(1-a_0x_0))^{\frac{1}{2}} B_n(x_0; a_0), \quad n \text{ even}$$

is an elliptic divisibility sequence over the ring \mathcal{R} . Every prime ideal \mathfrak{p} of \mathcal{R} is a divisor of (h) . Furthermore, if \mathfrak{p} does not divide both h_3 and h_4 then \mathfrak{p} has a unique rank of apparition ρ in (h) such that

$$(11.4) \quad h_n \equiv 0 \pmod{\mathfrak{p}} \quad \text{if and only if } n \equiv 0 \pmod{\rho}.$$

⁵ Two choices of u_0 are possible; for definiteness let u_0 be chosen with smallest imaginary part; if u_0 is real, with smallest real part.

If p does divide both h_3 and h_4 , it divides every subsequent term of (h) by Theorem 10.5. We call such primes "null divisors" of (h) .

12. Let p be a null divisor of (h) . Then

$$(12.1) \quad B_3(x_0; a_0) \equiv 0 \pmod{p}, \\ ((1-x_0)(1-a_0x_0))^{\frac{1}{2}} B_4(x_0; a_0) \equiv 0 \pmod{p}.$$

We first consider the case when p divides both B_3 and B_4 . Then the two polynomials $B_3(x, a_0)$ and $B_4(x, a_0)$ have a common root in the field \mathcal{R}_1/p . Hence their resultant must vanish in this field. Now this resultant is found to have the value $2^{20}a_0^4(1-a_0)^9$. Hence either $a_0 \equiv 0$, $a_0 \equiv 1$ or $2 \equiv 0 \pmod{p}$.

$a_0 \not\equiv 0 \pmod{p}$. For if $a_0 \equiv 0$, then by Table V, $B_3(x_0, a_0) \equiv 3 - 4x_0$ and $B_4(x_0, a_0) \equiv 4 - 8x_0 \pmod{p}$. But the congruences $3 - 4x_0 \equiv 4 - 8x_0 \equiv 0 \pmod{p}$ are impossible.

If $a_0 \equiv 1 \pmod{p}$, then $x_0 \equiv 1 \pmod{p}$. For by Tables IV and V,

$$B_3(x_0, a_0) \equiv (1-x_0)^3(3-x_0) \pmod{p} \\ B_4(x_0, a_0) \equiv 4(1-x_0)^5(1+x_0) \pmod{p}$$

if $a_0 \equiv 1$. Hence if $x_0 \not\equiv 1 \pmod{p}$, we must have $3 - x_0 \equiv 2 + 2x_0 \equiv 0 \pmod{p}$ but $1 - x_0 \not\equiv 0 \pmod{p}$, which is impossible.

Now finally if $2 \equiv 0 \pmod{p}$ but $a_0 \not\equiv 1 \pmod{p}$, since $B_4(x_0; a_0) \equiv 0 \pmod{2}$ and $B_3(x_0; a_0) \equiv (1 - a_0x_0^2)^2 \pmod{2}$ we must have $1 - a_0x_0^2 \equiv 0 \pmod{p}$. Hence since $a_0 \not\equiv 1 \pmod{p}$,

$$((1-x_0)(1-a_0x_0))^{\frac{1}{2}} \not\equiv 0 \pmod{p}.$$

If on the other hand $((1-x_0)(1-a_0x_0))^{\frac{1}{2}} \equiv 0 \pmod{p}$, it is easily shown that the previous case $a_0 \equiv x_0 \equiv 1 \pmod{p}$ must hold. We have thus proved

THEOREM 12.1. *The only prime ideal null divisors of (h) are primes dividing $2(1-a_0)$. Necessary and sufficient conditions that p be a null divisor are that either*

$$a_0 \equiv 1 \text{ and } x_0 \equiv 1 \pmod{p} \text{ or} \\ 2 \equiv 0 \text{ and } a_0x_0^2 \equiv 1 \pmod{p} \text{ but} \\ a_0 \not\equiv 1 \text{ and } x_0 \not\equiv 1 \pmod{p}.$$

In the lemniscate case, for example when $a_0 = -1$, the only null divisors are divisors of two for which $x_0 \equiv 1 \pmod{p}$. Hence if x_0 is an even rational integer, there are no null divisors.

13. We may classify the prime ideals of both \mathfrak{R} and \mathfrak{R}_1 into three categories:

- I. Ideals dividing $2a_0(1 - a_0)$.
- II. Ideals dividing $x_0(1 - x_0)(1 - a_0x_0)$.
- III. Ideals dividing neither $2a_0(1 - a_0)$ nor $x_0(1 - x_0)(1 - a_0x_0)$.

Ideals of the first and second categories will be called irregular; they include all null divisors, and are usually finite in number. Ideals of the third category will be called regular. In this section, we shall determine their laws of apparition in the elliptic sequences (A), (B), (C) and (D).

THEOREM 13.1. *No regular prime ideal can divide any two of A_n , B_n , C_n and D_n for the same value of n . Common divisors of A_n, B_n ; A_n, C_n ; \dots , C_n, D_n are always null divisors of the sequence (B).*

Proof. Suppose, for example, that for a certain fixed value of n

$$A_n \equiv 0 \pmod{p} \text{ and } B_n \equiv 0 \pmod{p}, \quad p \text{ regular.}$$

Then $K_{0n} \equiv 0 \pmod{p}$ and $K_{1n} \equiv 0 \pmod{p}$. Hence by the duplication formulas (5.3) and (5.4),

$$K_{12n} \equiv K_{11}K_{12n+1} \equiv 0 \pmod{p}$$

so that $K_{11}h_{2n} \equiv K_{11}^2h_{2n+1} \equiv 0 \pmod{p}$ by formula (11.1). But $K_{11}^2 = sn^2u_0 = x_0 \not\equiv 0 \pmod{p}$. Hence p divides two consecutive terms of the elliptic divisibility sequence (h). Therefore by Theorems 11.1, 10.5 and 12.2, $h_3 \equiv h_4 \equiv 0 \pmod{p}$, $2(1 - a_0) \equiv 0 \pmod{p}$, contrary to the hypothesis that p is regular.

It can be shown from the duplication formulas that a similar contradiction ensues if it is assumed that any other pair from A_n, B_n, C_n and D_n is divisible by p .

14. For regular prime ideals, the rank ρ of p in (h) and in (B) is evidently the same. Hence if p is regular,

$$(14.1) \quad B_n \equiv 0 \pmod{p} \quad \text{if and only if } n \equiv 0 \pmod{\rho},$$

where ρ is a fixed positive integer depending only on p and B_3 and B_4 .

THEOREM 14.1. *Let p be a regular prime ideal. Then if the rank of apparition of p in (B) is odd, p is not a divisor of (A), (C) or (D).*

Proof. Let p be regular. Then $B_n \equiv 0 \pmod{p}$ if and only if $K_{1n} \equiv 0 \pmod{p}$; similarly A_n, C_n or D_n are divisible by p only if K_{0n}, K_{2n} or K_{3n} are divisible by p . Suppose that p is of odd rank ρ in (B) and a divisor

of (A), for example. Then there exists a term A_k of (A) such that $A_k \equiv 0 \pmod{p}$. Now by the duplication formula (5.4),

$$(14.2) \quad K_{2k} \equiv 2K_{0k}K_{1k}K_{2k}K_{3k}.$$

Hence by the preceding remarks, $B_{2k} \equiv 0 \pmod{p}$ so that $\rho \mid 2k$ by (14.1). Therefore $\rho \mid k$ so that $A_k \equiv B_k \equiv 0 \pmod{p}$ contrary to Theorem 13.1. In like manner, p cannot be a divisor of (C) or (D).

THEOREM 14.2. *Let p be a regular prime ideal of even rank of apparition in (B). Then p is a divisor of precisely one of the three sequences (A), (C) and (D). If p is a divisor of (C), then $C_n \equiv 0 \pmod{p}$ if and only if n is an odd multiple of $\rho/2$, with similar results if p is a divisor of (A) or (D).*

Proof. Let p and ρ satisfy the hypothesis of the theorem. Then by the duplication formula (5.4)

$$K_{1\rho} \equiv 2K_{0(\rho/2)}K_{1(\rho/2)}K_{2(\rho/2)}K_{3(\rho/2)} \pmod{p}.$$

Hence since $B_\rho \equiv 0 \pmod{p}$ and p is regular, precisely one of $A_{\rho/2}$, $B_{\rho/2}$, $C_{\rho/2}$, $D_{\rho/2}$ is divisible by p . Evidently, $B_{\rho/2} \not\equiv 0 \pmod{p}$. Assume that $C_{\rho/2} \equiv 0 \pmod{p}$ so that p is a divisor of (C). Then p is not a divisor of (A) or (D). For if for example $D_k \equiv 0 \pmod{p}$, then by the duplication formula (14.2), $B_{2k} \equiv 0 \pmod{p}$. Hence $\rho \mid 2k$, $\rho/2 \mid k$ and $B_k \equiv 0 \pmod{p}$ contrary to Theorem 13.1. In precisely the same way we can show that $A_k \not\equiv 0 \pmod{p}$, and that if $C_k \equiv 0 \pmod{p}$, then k must be an odd multiple of $\rho/2$. It remains to prove the converse of this last statement. Consider then any term C_k of the sequence (C) in which k is a multiple of $\rho/2$. Then $2k$ is a multiple of ρ . Hence by (14.1) and (14.2)

$$0 \equiv 2A_kB_kC_kD_k \pmod{p}.$$

Now either B_k or C_k must be divisible by p but not both, by what we have already proved. If k is an even multiple of $\rho/2$, it is a multiple of ρ , so that $B_k \equiv 0$ and $C_k \not\equiv 0 \pmod{p}$. But if k is an odd multiple of $\rho/2$, it is not a multiple of ρ . Consequently, $B_k \not\equiv 0 \pmod{p}$, so that $C_k \equiv 0 \pmod{p}$, completing the proof for regular divisors of (C). The proof for divisors of (A) or (D) is precisely similar.

15. It remains to discuss the laws of apparition of the irregular prime ideals of categories I and II in Section 13. Since the elliptic polynomials have rational integral coefficients, if p is a prime ideal dividing a_0 say, we have $A_n(x; a_0) \equiv A_n(x; 0) \pmod{p}$. Consequently the arithmetical behavior of the sequences modulo p is given immediately by the algebraic behavior of the elliptic polynomials in the following five singular cases:

(15.1) (i) $a = 0$; (ii) $a = 1$; (iii) $x = 0$; (iv) $x = 1$; (v) $ax = 1$.

Here the first two cases apply to all prime ideals of category I save divisors of two, while the last three cases apply to prime ideals of category II. We discuss the former two cases in this section, and the latter three in Section 16.

Case 15.1 (i) $a = 0$.

Then $k^2 = 0$ and snu becomes $\sin u$, cnu becomes $\cos u$ and dnu becomes one. Thus if

$$U_n = U_n(x) = (\alpha^n - \beta^n)/(\alpha - \beta) \quad V_n = V_n(x) = \alpha^n + \beta^n$$

are the Lucas functions of the quadratic equation $t^2 - 2t\sqrt{1-x} + 1 = 0$ where

$$(15.2) \quad x = \sin^2 u,$$

then we readily find that

$$\begin{aligned} A_n(x; 0) &= D_n(x; 0) = 1; \\ B_n(x; 0) &= \sin nu / \sin u = U_n(x), & n \text{ odd,} \\ &= \sin nu / (\sin u \cos u) = 2U_n(x)/V_1(x), & n \text{ even;} \\ C_n(x; 0) &= \cos nu / \cos u = V_n(x)/V_1(x), & n \text{ odd} \\ &\cos nu = V_n(x)/2, & n \text{ even.} \end{aligned}$$

We thus obtain the following theorem:

THEOREM 15.1. If \mathfrak{p} is a prime ideal of the first category dividing a_0 , then

$$\begin{aligned} A_n &\equiv D_n \equiv 1 \pmod{\mathfrak{p}} \\ B_n &\equiv U_n, \quad n \text{ odd}; \equiv 2U_n/V_1, \quad n \text{ even}; \\ C_n &\equiv V_n/V_1, \quad n \text{ odd}; \equiv \frac{1}{2}V_n, \quad n \text{ even.} \end{aligned}$$

Here $U_n = U_n(x_0)$ and $V_n = V_n(x_0)$ are the Lucas functions of the quadratic equation $t^2 - 2(1 - x_0)^{1/2}t + 1 = 0$.

Case 15.1 (ii). $a = 1$.

Then $k^2 = 1$ and snu becomes $\tanh u$, while cnu and dnu become $\operatorname{sech} u$. Now by the transformation formulas of Table IV,

$$\begin{aligned} A_n(x; 1) &= (1-x)^{an} C_n(x/(x-1); 0) \\ B_n(x; 1) &= (1-x)^{\beta n} B_n(x/(x-1); 0) \\ C_n(x; 1) &= (1-x)^{an} A_n(x/(x-1); 0) \\ D_n(x; 1) &= (1-x)^{an} D_n(x/(x-1); 0). \end{aligned}$$

Hence if we let $u = ir$, then $x = -\tan^2 r$ and $x/(x-1) = \sin^2 r$. Therefore, by the results of case (i),

$$\begin{aligned} A_n(x; 1) &= \sec r^{n^2} \cos nr = (2/V_1)^{n^2} V_n/2^6 \\ &= \sec r^{n^2-3} (\sin nr/\sin r) = (2/V_1)^{n^2-3} U_n, & n \text{ even} \\ B_n(x; 1) &= \sec r^{n^2-1} (\sin nr/\sin r) = (2/V_1)^{n^2-1} U_n, & n \text{ odd} \\ C_n(x; 1) &= D_n(x; 1) = \sec r^{2an} = (2/V_1)^{2an}. \end{aligned}$$

Here $U_n = U_n(x/(x-1))$, $V_n = V_n(x/(x-1))$ are the Lucas functions of the quadratic equation $t^2 - 2t/(1-x)^{\frac{1}{2}} + 1 = 0$.

We thus obtain the following theorem:

THEOREM 15.2. *If \mathfrak{p} is a prime ideal of the first category dividing $1 - a_0$, then*

$$\begin{aligned} A_n &\equiv (2/V_1)^{n^2} V_n/2; \\ B_n &\equiv \begin{aligned} &(2/V_1)^{n^2-1} U_n, & n \text{ odd,} \\ &(2/V_1)^{n^2-3} U_n, & n \text{ even;} \end{aligned} \\ C_n &\equiv D_n \equiv \begin{aligned} &(2/V_1)^{n^2-1}, & n \text{ odd,} \\ &(2/V_1)^{n^2}, & n \text{ even.} \end{aligned} \end{aligned}$$

Here $U_n = U_n(x_0/(x-1))$, $V_n = V_n(x_0/(x_0-1))$ are the Lucas functions of the quadratic equation $t^2 - 2t/(1-x_0)^{\frac{1}{2}} + 1 = 0$.

For prime ideals of the first category dividing two, a special discussion must be made as in the case of the rational field for the prime two treated in M, pages 40-41. We shall not pursue the matter further here.

16. Consider now prime ideals of the second category. We may confine ourselves to ideals which are not also of the first category; for ideals of both categories are either null divisors of (B) or else are already covered by the results of Section 15. A prime ideal of this character is easily shown to divide precisely one of the algebraic numbers x_0 , $1 - x_0$ or $1 - a_0 x_0$. Clearly then $A_n(x_0, a_0)$ is congruent to either $A_n(0, a_0)$, $A_n(1, a_0)$ or $A_n(1/a_0, a_0)$, with similar results for B_n , C_n and D_n .

By the results of Table IV, we find that

⁶ For example, $A_3(x; 1) = (1-x^3)(1+3x)$ by direct calculation from Table V. On the other hand, the formula for $n=3$ becomes $A_3(x; 1) = \sec r^9 (\cos 3r/\cos r)$. Now $\cos 3r/\cos r = 4 \cos^2 r - 3 = [1 + 3(-\tan^2 r)]/\sec^2 r$ and $\sec^2 r = 1 - (-\tan^2 r)$. Hence $A_3(x; 1) = (1 - (-\tan^2 r))^3 (1 + 3(-\tan^2 r))$, checking, since $x = -\tan^2 r$.

	n odd	n even
$A_n(0) =$	1	1
$B_n(0) =$	n	n
$C_n(0) =$	1	1
$D_n(0) =$	1	1
$A_n(1) =$	$(1-a)^{(n^2-1)/4}$	$(1-a)^{n^2/4}$
$B_n(1) =$	$(-1)^{(n-1)/2}(1-a)^{(n^2-1)/4}$	$(-1)^{n/2}(1-a)^{(n^2-4)/4}$
$C_n(1) =$	$(-1)^{(n-1)/2}(1-a)^{(n^2-1)/4}n$	$(-1)^{n/2}(1-a)^{n/4}$
$D_n(1) =$	$(1-a)^{(n^2-1)/4}$	$(1-a)^{n^2/4}$
$A_n(1/a) =$	$((1-a)/a)^{(n^2-1)/4}$	$(-1)^{n/2}((1-a)/a)^{n^2/4}$
$B_n(1/a) =$	$(-1)^{(n-1)/2}((1-a)/a)^{(n^2-1)/4}$	$((1-a)/a)^{(n^2-4)/4}n$
$C_n(1/a) =$	$((1-a)/a)^{(n^2-1)/4}$	$(-1)^{n/2}((1-a)/a)^{n^2/4}$
$D_n(1/a) =$	$(-1)^{(n-1)/2}((1-a)/a)^{(n^2-1)/4}n$	$((1-a)/a)^{(n^2-4)/4}$

We deduce the following theorems from these results.

THEOREM 16.1. *Let \mathfrak{p} be a prime ideal of the second category which is not of the first category. Then \mathfrak{p} never divides the sequence (A) . Furthermore \mathfrak{p} divides (B) , (C) or (D) according as $x_0 \equiv 0$, $1 - x_0 \equiv 0$ or $1 - a_0x_0 \equiv 0 \pmod{\mathfrak{p}}$. Its rank of apparition in every case is p , where p is the rational prime which \mathfrak{p} divides.*

THEOREM 16.2. *Under the hypotheses of the preceding theorem, if \mathfrak{p} divides (C) , it does not divide (D) , and its rank of apparition in (B) is $2p$. Furthermore, $C_n \equiv 0 \pmod{\mathfrak{p}}$ if and only if n is an odd multiple of p . Similar results hold for divisors of (D) .*

17. If we compare the results of Sections 15 and 16 for irregular prime ideals, we see that the laws of apparition are the same for all ideals save null divisors, and that the laws of apparition for (A) , (C) and (D) essentially generalize Lucas' law of apparition for (V) , and in a sense explain it. Furthermore, the laws of apparition for ideals of the field \mathfrak{F}_1 are precisely the same as for the field \mathfrak{F} . If in particular then x_0 and a_0 are rational integers, the four elliptic sequences are sequences of rational integers, and the ideals become ordinary primes. It is of some interest to note that the Lucas sequences associated with the primes of the first category in this case involve quadratic irrationalities, and are of the type studied in Lehmer's thesis. (D. H. Lehmer [1])

VI. Conclusion. The Laws of Repetition.

18. We conclude the paper by giving the laws of repetition for powers of primes in rational integral elliptic sequences. The extension to arbitrary algebraic integral sequences is easy, but will not be discussed here.

Assume then that x_0 and a_0 are rational integers. We need consider only regular primes p not dividing $2a_0(1-a_0)x_0(1-x_0)(1-a_0x_0)$. Let p be such a prime. Then p is odd. Assume that p divides (D) , and that its rank in (B) is 2ρ . Then the rank of p in (D) is ρ , and p does not divide (A) or (C) , by the results of Chapter IV. Consequently by the duplication formula (5.4) if p^k is the highest power of p dividing $B_{2\rho}$, it is also the highest power of p dividing D_ρ . Now since (B) is essentially the elliptic divisibility sequence (h) so far as regular primes are concerned, the law of repetition of powers of p in (B) follows from the results in Ward [2] for elliptic divisibility sequences; namely, if $l \leq k$, the rank of apparition of p^l in (B) is 2ρ , and if $l \geq k$, the rank is $2p^{l-k}\rho$. Now p is odd, and the only terms of (D) divisible by p are odd multiples of ρ . Hence since $D_n \equiv 0 \pmod{p^l}$ if and only if $B_{2n} \equiv 0 \pmod{p^l}$, we can state the following theorem:

THEOREM. *Let x_0 and a_0 be rational integers, and p a regular prime of rank ρ in (D) . Furthermore, let p^k be the highest power of p dividing D_ρ . Then the rank of apparition ρ^* of p^l in (D) is ρ or $p^{l-k}\rho$ according as $l \leq k$ or $l \geq k$. Furthermore $D_n \equiv 0 \pmod{p^l}$ if and only if n is an odd multiple of ρ^* .*

Precisely similar results hold for prime divisors of (A) or (C) . For the Lucas functions, these results become Lucas' law of repetition for primes in (V) .

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HARMONIC DOUBLE SERIES.*

By LEONARD TORNHEIM.

The value of the sum c_n of the harmonic series in one variable $c_n = \sum_{p=1}^{\infty} p^{-n}$ is not known for n odd. For n even, $c_{2m} = 2^{2m-1} \pi^{2m} B_m / m!$, where B_m is the m -th Bernoulli number; in particular $c_2 = \pi^2/6$, $c_4 = \pi^4/90$, $c_6 = \pi^6/945$. We call

$$\sum_{p,q=1}^{\infty} p^{-r} q^{-s} (p+q)^{-t}$$

a harmonic double series. Its value will be denoted by (r, s, t) and we shall show how to express it as a polynomial in the c_n with rational coefficients for certain values of r, s, t .

The following lemma will be useful.

LEMMA 1. Let $f(p)$ be a monotone decreasing function with $\lim_{p \rightarrow \infty} f(p) = c$.

Suppose

$$\Phi(p, q; f) = f(q)/p(p+q) + f(p)/q(p+q) - f(p+q)/pq.$$

Then

$$(1) \quad \sum_{p,q=1}^{\infty} \Phi(p, q; f) = 2 \sum_{r=1}^{\infty} [f(r) - c]/r^2,$$

$$(2) \quad \sum_{\substack{p,q=1 \\ (p,q)=1}}^{\infty} \Phi(p, q; f) = 2[f(1) - c].$$

The terms in the series (1) and (2) are non-negative. For, by the monotonicity of $f(p)$

$$\Phi(p, q; f) \geq f(p+q) [1/p(p+q) + 1/q(p+q) - 1/pq] = 0.$$

Let $g(x) = f(x) - c$ if $c \neq -\infty$. (The proof of the lemma for the case $c = -\infty$ is almost obvious and will not be discussed.) Then $\Phi(p, q; g) = \Phi(p, q; f)$, and the proof may be restricted to the case $c = 0$.

$$\text{Since } 1/p - 1/(p+q) = q/(p+q)p,$$

* Received February 9, 1949.

$$\begin{aligned}\sum_{p+q < N} f(q)/p(p+q) &= \sum_{q < N-1} (f(q)/q) \sum_{p < N-q} (1/p - 1/(p+q)) \\ &= \sum_{q < N-1} (f(q)/q) \sum_{s=1}^q (1/s - 1/(N-s)).\end{aligned}$$

The last equality is obtained by noticing that

$$\sum_{s=1}^{N-q-1} 1/s + \sum_{s=N-q}^{N-1} 1/s = \sum_{s=1}^q 1/s + \sum_{s=q+1}^{N-1} 1/s;$$

thus

$$\begin{aligned}\sum_{p=1}^{N-q-1} (1/p - 1/(p+q)) &= \sum_{s=1}^{N-q-1} 1/s - \sum_{s=q+1}^{N-1} 1/s \\ &= \sum_{s=1}^q 1/s - \sum_{s=N-q}^{N-1} 1/s = \sum_{s=1}^q (1/s - 1/(N-s)).\end{aligned}$$

Next

$$\begin{aligned}\sum_{p+q < N} f(p+q)/pq &= \sum_{r=2}^{N-1} \sum_{q=1}^{r-1} f(r)/(r-q)q \\ &= \sum_r (f(r)/r) \sum_q (1/q + 1/(r-q)) \\ &= \sum_{r=2}^{N-1} (f(r)/r) 2 \sum_{q=1}^{r-1} 1/q;\end{aligned}$$

thus

$$\begin{aligned}\sum_{p+q < N} \Phi(p, q; f) &= 2 \sum_{q < N-1} (f(q)/q) \sum_{s=1}^q (1/s - 1/(N-s)) \\ &\quad - 2 \sum_{2 \leq q < N} (f(q)/q) \sum_{s=1}^{q-1} 1/s \\ &= 2 \sum_{q < N-1} (f(q)/q) (1/q) - 2f(N-1)/(N-1) \sum_{s=1}^{N-2} 1/s \\ &\quad - 2 \sum_{q < N-1} (f(q)/q) \sum_{s=1}^q 1/(N-s).\end{aligned}$$

It remains to be shown that the last two terms approach 0 with increasing N . The absolute value of the middle term is at most

$$(2f(N-1)/(N-1)) \{1 + \int_1^{N-1} (1/t) dt\} \rightarrow 0.$$

Half of the last term can be decomposed into two summations,

$$- \sum_{q < N-1} = - \sum_{q < q_0} - \sum_{q=q_0}^{N-2}$$

of the summands $(f(q)/q) \sum_{s=1}^q (1/(N-s))$. Choose q_0 large enough so that $f(q_0) < \epsilon$, and take N greater than $2q_0^2$. Then

$$\sum_{q < q_0} \leq f(1) \sum_{q < q_0} \frac{1}{q} \sum_{s=1}^q \frac{1}{(N/2)} \leq f(1)/q_0.$$

Finally

$$\begin{aligned} \sum_{q=q_0}^{N-2} &\leq \sum_q (f(q_0)/q) \int_1^{q+1} ds/(N-s) \\ &= -f(q_0) \sum \frac{1}{q} \log (N-1-q)/(N-1) \\ &\leq f(q_0) \int_{q_0}^{N-1} \{1/(N-1) + q/2(N-1)^2 + q^2/3(N-1)^3 + \dots\} dq \\ &\leq c_2 f(q_0). \end{aligned}$$

This completes the proof of the lemma.

Equation (2) may be proved similarly.¹ Also (1) can be derived from (2) by replacing p by λp , q by λq , and taking $\sum_{\lambda=1}^{\infty}$.

Every function $h(p)$ defined for positive integral p is a sum of two monotone functions h_i and h_d where

$$h_i(p) = \frac{1}{2}h(1) + \sum_{s=2}^p \max(0, h(s) - h(s-1)),$$

$$h_d(p) = \frac{1}{2}h(1) + \sum_{s=2}^p \min(0, h(s) - h(s-1)).$$

If either h_i or h_d is bounded then Lemma 1 is true for h replacing f .

In (1) take $f(p) = p^{-1}$ to obtain

COROLLARY 1. $(1, 1, 1) = 2c_3$.

Also

$$\sum_{(p,q)=1} \frac{1}{[pq(p+q)]} = 2.$$

In fact, $\sum_{\substack{p,q=1 \\ (p,q)=1}}^{\infty} p^{-r} q^{-s} (p+q)^{-t} = (r, s, t)/c_{r+s+t}.$

Setting $f(p) = p^{-n+2}$ in (1) yields the following corollary after using Lemma 2 below to assure the absolute convergence of the series. This property is needed to allow the rearrangement of the terms.

COROLLARY 2. If $n > 2$,

$$(3) \quad 2(n-2, 1, 1) - (1, 1, n-2) = 2c_n.$$

We write $[r, s, t]$ for $p^{-r} q^{-s} (p+q)^{-t}$; then $(r, s, t) = \sum_{p,q=1}^{\infty} [r, s, t].$

¹ This result was originally obtained geometrically by considering the areas of the triangles described by means of a diagram of Ford, "Fractions," *American Mathematical Monthly*, vol. 45 (1938), pp. 586-601. The triangles have vertices at the points $(p/q, f(q))$ where $0 \leq p/q \leq 1$ and $(p, q) = 1$, two vertices belonging to the same triangle if $|p_1 q_2 - p_2 q_1| = 1$. The proof given here was found jointly with N. J. Fine.

The solutions to be obtained will be based only on (3) and the following simple relationships

$$(4) \quad (r, s-1, t+1) + (r-1, s, t+1) = (r, s, t)$$

resulting from the fact that the same expression in brackets is an identity;

$$(5) \quad (r, s, t) = (s, r, t);$$

$$(6) \quad (r, s, 0) = c_r c_s;$$

and

$$(7) \quad (r, 0, s) + (s, 0, r) = c_r c_s - c_{r+s},$$

which comes from the fact that

$$\begin{aligned} \sum p^{-r}(p+q)^{-s} + \sum p^{-s}(p+q)^{-r} &= \sum_{u > p} p^{-r} u^{-s} + \sum_{u < p} u^{-s} p^{-r} \\ &= \sum_{p, u=1}^{\infty} p^{-r} u^{-s} - \sum_{u=1}^{\infty} u^{-r-s}. \end{aligned}$$

LEMMA 2. A necessary and sufficient condition that (r, s, t) be finite is that $r+t > 1$, $s+t > 1$, and $r+s+t > 2$.

Proof. Observe that

$$\begin{aligned} (r, s, t) &> \sum_{q < p} \{(p+q)/k_r\}^{-r} q^{-s} (p+q)^{-t} \\ &> k_r^r \sum_{p=2}^{\infty} \sum_{q=1}^{p-1} q^{-s} (k_{-r-t} p)^{-r-t} \left(k_t = \begin{cases} 1 & (i \geq 0) \\ 2 & (i < 0) \end{cases} \right) \end{aligned}$$

which is ∞ if $r+t \leq 1$. Otherwise it is greater than

$$k_r^r k_{-r-t}^{-r-t} \sum_{p=2}^{\infty} \int_1^{p/2} q^{-s} p^{-r-t} dq,$$

which is ∞ if $r+s+t \leq 2$. On the other hand if the conditions of the lemma are satisfied,

$$\begin{aligned} (r, s, t) &< \sum_{p \geq q} p^{-r} q^{-s} (k_t p)^{-t} + \sum_{q \geq p} p^{-r} q^{-s} (k_t q)^{-t} \\ &< k_t^{-t} \{c_{r+t} + c_{s+t} + \sum_{q=2}^{\infty} \int_{q-1}^{\infty} q^{-s} p^{-r-t} dp + \sum_{p=2}^{\infty} \int_{p-1}^{\infty} p^{-r} q^{-s-t} dq\} \end{aligned}$$

and this last expression is finite.

We shall assume that r, s, t in (r, s, t) are all integral. If every (r, s, t) is known for r, s, t all non-negative and $r+s+t = n$, by means of (4), (5),

and Lemma 2 it is possible to evaluate (r_1, s_1, t_1) for any integral r_1, s_1, t_1 with $r_1 + s_1 + t_1 = n$. Hereafter we shall consider only non-negative integral r, s, t . In view of (5) we shall usually state our results with $r \geq s$.

THEOREM 1.

$$(n-1, 1, 0) = \infty, \quad (n-1, 0, 1) = \infty, \quad (n, 0, 0) = \infty.$$

These follow from Lemma 2.

THEOREM 2. *If $n \geq 3$,*

$$(8) \quad (1, 1, n-2) = (n-1)c_n - \sum_{a=2}^{n-2} c_a c_{n-a},$$

$$(9) \quad (n-2, 1, 1) = \frac{1}{2}\{(n+1)c_n - \sum_{a=2}^{n-2} c_a c_{n-a}\},$$

$$(10) \quad (1, 0, n-1) = \frac{1}{2}\{(n-1)c_n - \sum_{a=2}^{n-2} c_a c_{n-a}\}.$$

First from (4), $(a, 0, n-a) + (a-1, 1, n-a) = (a, 1, n-a-1)$ so that summing on a from 2 to $n-2$,

$$\sum_{a=2}^{n-2} (a, 0, n-a) + (1, 1, n-2) = (n-2, 1, 1).$$

But from (7)

$$2 \sum_{a=2}^{n-2} (a, 0, n-a) = \sum_{a=2}^{n-2} c_a c_{n-a} - (n-3)c_n,$$

so that

$$\sum_{a=2}^{n-2} c_a c_{n-a} - (n-3)c_n + 2(1, 1, n-2) = 2(n-2, 1, 1).$$

Using this result simultaneously with the equation of Corollary 2 of Lemma 1, the first two equations of the theorem are readily established. Equation (10) follows from (4), (5), and (8).

THEOREM 3.

$$(11) \quad (a, n-a-1, 1) = (-1)^a \left\{ \sum_{i=2}^a (-1)^i c_i c_{n-i} + \frac{1}{2} \sum_{j=2}^{n-2} c_j c_{n-j} - \frac{1}{2}(n+1)c_n \right\}.$$

All that is needed to prove this are equations (9), (5), and the equation

$$(a, n-a-1, 1) + (a-1, n-a, 1) = (a, n-a, 0) = c_a c_{n-a},$$

this following directly from (4) and (6).

COROLLARY. If n is even, i. e., $n = 2m$,

$$\sum_{k=1}^{m-1} c_{2k} c_{n-2k} = \frac{1}{2}(n+1)c_n.$$

It is a consequence of the fact that $(a, n-a-1, 1) = (n-a-1, a, 1)$. This identity can be translated into one for Bernoulli numbers.

Hereafter $[m]$ means the largest integer not exceeding m .

THEOREM 4. For n odd, i. e., $n = 2m+1$, and $a > n/2$,

$$\begin{aligned} (a, n-a-2, 2) &= (-1)^a (2a-2m+1) \left\{ \sum_{k=1}^{[m/2]} c_{2k} c_{n-2k} - \frac{1}{4}(n+1)c_n \right\} \\ &\quad + (-1)^a \sum_{i=m+1}^a (-1)^i (a-i+1) c_i c_{n-i}. \end{aligned}$$

Proof. From

$$(j, n-j-2, 2) = (j, n-j-1, 1) - (j-1, n-j-1, 2)$$

by summing get

$$(a, n-a-2, 2) = \sum_{j=m+1}^a (-1)^{j-a} (j, n-j-1, 1) + (-1)^{m-a} (m, m-1, 2).$$

Use the previous theorem, and the fact that $(m, m-1, 2) = \frac{1}{2}(m, m, 1)$ by equation (4), and also that $\sum_{i=2}^{n-2} c_i c_{n-i} = 2 \sum_{i=2}^m c_i c_{n-i}$.

THEOREM 5. If $n > 1$, $(0, 0, n) = c_{n-1} - c_n$.

$$\text{Proof. } \sum_{p,q} 1/(p+q)^n = \sum_{r=1}^{\infty} \sum_{p+q=r} 1/r^n = \sum_r (r-1)/r^n = c_{n-1} - c_n.$$

THEOREM 6. For n even, i. e., $n = 2m$, $(m, 0, m) = \frac{1}{2}(c_m^2 - c_{2m})$.

Proof. Immediate from (7).

Denote $(a, 0, n-a)$ by f_a . Every quantity (r, s, t) for fixed $n = r + s + t$ and r, s, t all non-negative integers may be expressed as a linear combination of the f_a ($a = 0, \dots, n$). For, if $r \neq 0 \neq s$, then by (4) (r, s, t) may be expressed in terms of similar expressions having larger t . This process is repeated, if necessary, until only expressions (r', s', t') are obtained with either $r' = 0$ or $s' = 0$. In the next lemma the f_a are expressed in terms of the quantities $d_i = (i, i, n-2i)$ ($i = 1, \dots, [n/2] - 1$) and polynomials in the c_j . Hence (r, s, t) may be expressed in the same way.

Using the explicit formulas of the next lemma and the relation (7), we obtain linear equations for the d_i which are solved in certain cases to give finally Theorems 7 and 8.

LEMMA 3. $(a, 0, n-a) = \frac{1}{2}M_a + L_a$ ($a = 2, \dots, n-2$), where

$$M_a = \sum_{j=1}^{\min(a,m)} (-1)^{a-j} \{ {}_jC_{a-j} + {}_{j-1}C_{a-j-1} \} d_j$$

and L_a is a polynomial in c_2, \dots, c_n , and $m = [n/2]$.

Denote $[a, b, c] + [b, a, c]$ by $[a, b, c]'$. We shall prove the corresponding identity in p, q when $[a, b, c]'$ replaces $2(a, b, c)$ and L_a is a linear combination of $[j, n-j-1, 1]$ and $[j, n-j, 0]$. Then summation over all positive p, q followed by the use of (11) and (6) will yield the lemma.

The proof will be by an induction on a . For $a = 2$, we have

$$[2, 0, n-2]' = \frac{1}{2} \{ [2, 2, n-4]' - 2[1, 1, n-2]' \},$$

the validity of which can be easily verified. Likewise the identity holds for $a = 3$:

$$[3, 0, n-3]' = \frac{1}{2} \{ [3, 3, n-6]' - 3[2, 2, n-4]' \}.$$

Next, to make the induction step, assume the result for a and $a-1$, and prove it for $a+1$. Multiply both sides of the identities for a and $a-1$ by $[1, 1, -2]$ except for terms of the form $[r, s, 0]$ or $[r, s, 1]$, these terms being multiplied by

$$(11.5) \quad [1, -1, 0] + 2[0, 0, 0] + [-1, 1, 0]$$

which equals $[1, 1, -2]$, and then subtract. An application of (4) to the left hand side and a simple manipulation of the other side using the identity ${}_rC_s + {}_rC_{s-1} = {}_{r+1}C_s$ proves the identity in the summands $[r, s, t]$.

We must show finally that no term (r, s, t) in L_a is infinite; i.e., if $t = 1$ then $r > 0$ and $s > 0$, while if $t = 0$ then $r > 1$ and $s > 1$. But the only d_i having $t = 0$ or 1 is $d_m = (m, m, t)$ and it first occurs in the induction when $a = m$. Each subsequent multiplication by (11.5) reduces r or s by 1. Only $n-m-2$ more steps are required to reach the last step when $a = n-2$, and r and s then still satisfy the required conditions. The proof of the lemma is complete.

When the equations of the lemma are combined with (7), $m-1$ equations in the $m-2$ unknowns d_j are obtained. Write $N_a = M_a + M_{n-a}$; then N_a is known in terms of c_2, \dots, c_n by Lemma 3 and equation (7). A relationship of dependence is easily noticed by adding the N_a ; from Lemma 3 the sum is simply zero.

Ultimately we shall have to work with N_a since its value in terms of the c_j 's is known. But it is easier to treat M_a and M_{n-a} separately and then add later.

We shall next show that for $a \leq m$,

$$(12) \quad d_a = 2 \sum_{i=1}^a {}_{2a-i-1}C_{a-1} (i, 0, n-i) \quad (a = 2, \dots, m).$$

This is in fact true for the corresponding summands

$$(13) \quad [a, a, n-2a] = \sum_{i=1}^a {}_{2a-i-1}C_{a-1} \{[i, 0, n-i] + [0, i, n-i]\}.$$

We proceed by induction, equation (13) being true if $a = 1$. Multiply both sides of (13) by $[1, 1, -2]$. On the right are obtained terms $[n+1, 1, *]$. (In the notation $(b, c, *)$ the value of $*$ is to be chosen so that $b + c + * = n$.) But since (4) is true if brackets replace the parentheses,

$$[i+1, 1, *] = \sum_{j=1}^{i+1} [j, 0, *] + [0, 1, *];$$

and similarly for $[1, i+1, *]$. Hence, assuming (13),

$$\begin{aligned} [a+1, a+1, *] &= \sum_{i=1}^a {}_{2a-i-1}C_{a-1} \left\{ \sum_{j=1}^{i+1} [j, 0, *]' + [0, 1, *]' \right\} \\ &= \sum_{j=2}^{a+1} \sum_{i=j-1}^a {}_{2a-i-1}C_{a-1} [j, 0, *]' + 2 \sum_{i=1}^a {}_{2a-i-1}C_{a-1} [1, 0, *]'. \end{aligned}$$

But $kC_a + kC_{a-1} = {}_{k+1}C_a$, so that

$$\begin{aligned} [a+1, a+1, *] &= \sum_{j=2}^{a+1} {}_{2a+1-j}C_a [j, 0, *]' + (2a/a) {}_{2a-1}C_a [1, 0, *]' \\ &= \sum_{j=1}^{a+1} {}_{2(a+1)-j-1}C_{(a+1)-1} [j, 0, *]', \end{aligned}$$

as was to be proved.

We next take the corresponding sums for M_{n-i} ,

$$\sum_{i=1}^a {}_{2a-i-1}C_{a-1} M_{n-i} = \sum_{i=1}^a {}_{2a-i-1}C_{a-1} \sum_{j=1}^m (-1)^{n-i-j} \{ {}_jC_{n-i-j} + {}_{j-1}C_{n-i-j-1} \} d_j$$

where $a \leq m$ and prove

$$\text{LEMMA 4.} \quad \sum_{i=1}^a {}_{2a-i-1}C_{a-1} M_{n-i} = \sum_{j=1}^m b_{aj} d_j \quad (a = 2, \dots, m)$$

where $b_{aj} = 0$ if

$$(14) \quad a < n - 2j,$$

or

$$(15) \quad j > a \text{ and } j > n - 2a.$$

The coefficient of d_j is

$$(16) \quad b_{aj} = \sum_{i=1}^a {}_{2a-i-1}C_{a-1} (-1)^{n-i-j} \{ {}_jC_{n-i-j} + {}_{j-1}C_{n-i-j-1} \} \\ = \frac{(j-1)!}{(a-1)!} \sum_{i=n-2j}^a (-1)^{n-i-j} \frac{(2a-i-1)!(n-i)}{(n-i-j)!(2j+i+n)!(a-i)!}$$

and the factor involving the summation sign becomes, on setting $k = n - i - j$,

$$(17) \quad \sum_{k=k_0}^j (-1)^k \frac{(2a+k+j-n-1)!(k+j)}{k!(j-k)!(a+j-n+k)!},$$

where $k_0 = \max(n - a - j, 0)$. Now since $2a + k + j - n - 1 \geq a + j - n + k$ we have that

$$(18) \quad \frac{(2a+k+j-n-1)!(k+j)}{(a+j-n+k)!} \\ = (2a+k+j-n-1)(2a+k+j-n-2) \\ \cdots (a+j-n+k+1)(k+j)$$

and the right hand side considered as a polynomial in k has degree a and is expressible as

$$(19) \quad \sum_{l=0}^a A_l k!/(k-l)!$$

where the A_l are constants, and $k!/(k-l)! = k(k-1) \cdots (k-l+1)$ if $l \geq 1$ and 1 if $l = 0$.

Setting $k = 0, 1, 2, \dots$ in turn we see that

$$(20) \quad A_l = 0 \quad (n+1-2a-j \leq 0 \leq l \leq n-a-j-1).$$

Then (17) becomes

$$(21) \quad \sum_{k=k_0}^j \sum_{l=0}^a (-1)^k A_l / (j-k)!(k-l)! \\ = \sum_{l=0}^a \frac{A_l}{(j-l)!} \sum_{k=k_l}^j (-1)^k \frac{(j-l)!}{(j-k)!(k-l)!},$$

where $k_l = \max(n - a - j, l)$. Here we have taken $k \geq l$ since $k!/(k-l)! = k(k-1) \cdots (k-l+1)$ and is zero if $0 \leq k < l$ (see (19)). Thus the left hand side of (21) equals

$$(22) \quad \sum_{l=0}^a \frac{A_l}{(j-l)!} \sum_{k=k_l}^j (-1)^k {}_{j-l}C_{j-k}.$$

We know from equation (16) that b_{aj} is zero if (14) holds. We want to get the conditions of (15).

By (20), $A_l = 0$ for $0 \leq l \leq n - a - j - 1$ if $n + 1 - 2a - j \leq 0$ or

$$(23) \quad j > n - 2a.$$

Also

$$\sum_{k=l}^j (-1)^k {}^{j-l}C_{j-k} \quad (l \geq n - a - j)$$

will equal zero unless $j = l$ since it is the binomial expansion of $(-1)^l (1 - 1)^{j-l}$. But $j \neq l$ if

$$(24) \quad j > a$$

since $l \leq a$. Thus Lemma 4 is proved.

We wish to compute the values of certain coefficients which are not zero.

LEMMA 5. If $a = n - 2j$, then $b_{aj} = (-1)^{j/2}$. If $a > n - 2a$, then $b_{aa} = (-1)^a$. Finally if $j > a$ and $j = n - 2a$, then $b_{aj} = (-1)^a$.

If in (16) we take $a = n - 2j$, we get

$$b_{n-2j,j} = {}_{a-1}C_{a-1} (-1)^{n-n+2j-j} \{ {}_jC_{n-n+2j-j} + {}_{j-1}C_{j-1} \} = (-1)^{j/2}.$$

If (23) holds, but $j = a$ (instead of (24)), then by the reasoning preceding (24), (22) becomes $(-1)^a A_a$ since all terms are zero except when $l = a$. But by (19), A_a is the coefficient of k^a in (18), so that $A_a = 1$. Then from (16) we find $b_{aa} = (-1)^a$.

Finally if $j > a$ and instead of (23), $j = n - 2a$, then (17) becomes

$$\sum_{k=a}^{n-2a} (-1)^k (k + n - 2a) / k (n - 2a - k)! (k - a)!.$$

Denote this sum by S_n . First $S_{3a+1} = (-1)^a/a$. Next

$$\begin{aligned} S_n &= \sum_{k=a}^{n-2a} (-1)^k \{ 1/(n - 2a - k)! (k - a)! \\ &\quad + (n - 2a)/k (n - 2a - k)! (k - a)! \} \\ (25) \quad &= (-1)^a (1 - 1)^{n-3a} / (n - 3a)! \\ &\quad + (n - 2a) \sum_k (-1)^k / k (n - 2a - k)! (k - a)!. \end{aligned}$$

Thus if $n - 3a > 0$,

$$\begin{aligned} S_n &= 0 + \sum_{k=a}^{n-2a} (-1)^k \{ (n - 2a - k)/k (n - 2a - k)! (k - a)! \\ &\quad + 1/(n - 2a - k)! (k - a)! \} \\ &= \{ \sum_{k=a}^{n-1-2a} (-1)^k / k (n - 1 - 2a - k)! (k - a)! \} \\ &\quad + (-1)^a (1 - 1)^{n-3a} / (n - 3a)! = S_{n-1} / (n - 2a - 1) \end{aligned}$$

because of the last part of (25).

We conclude that the coefficient of d_j is, when $j = n - 2a$,

$$\begin{aligned} ((j-1)!/(a-1)!)S_n &= ((n-2a-1)!/(a-1)!)S_n \\ &= ((n-2a-2)!/(a-1)!)S_{n-1} \\ &= \cdots = (a!/(a-1)!)S_{a+1} = a(-1)^a/a = (-1)^a. \end{aligned}$$

Since $(a, 0, n-a) + (n-a, 0, a)$ is known by (7), we obtain from (12) and Lemmas 3 and 4 the following simultaneous linear equations in the d_a :

$$(26) \quad d_a + \sum_j b_{aj}d_j = g_a,$$

where the b_{aj} are known constants, j ranges from $[(n-a+1)/2]$ to m , and g_a are polynomials in the c_i with rational coefficients.

THEOREM 7. *If n is odd, $(r, s, n-r-s)$ may be explicitly determined as a polynomial in c_2, \dots, c_n with rational coefficients.*

We shall treat only the case $n \equiv 1 \pmod{6}$ i. e., $n = 6r + 1$. Separate the equations of (26) into two groups according as $a \leq 2r$ or not. In the first case $b_{aj} = 0$ if $j \geq n + 1 - 2a$ from (15), and (26) need be summed only to $n - 2a$.

Also $b_{aj} = 0$ if simultaneously

$$(27) \quad j \geq n + 1 - 2a,$$

$$(28) \quad j > a.$$

If $a \leq 2r$ then (27) implies (28), for $j > n - 2a \geq 6r + 1 - 4r > 2r \geq a$, whereas if $a > 2r$ then (27) is a consequence of (28), for $j \geq a + 1 \geq 2r + 2 = (6r + 1) + 1 - 2(2r) > n + 1 - 2a$.

To solve the system (26) first take $a = 2r + 1$. The summation in (26) is from $2r$ to $2r + 1$, and by Lemma 5 the left hand side of this equation reduces to $2d_{2r}$; thus $d_{2r} = \frac{1}{2}g_{2r+1}$. To find the other d_a we divide all equations (26) into three sets of equations

$$\begin{array}{ll} \text{I:} & a = 2r + 2s \quad (s = 1, 2, \dots) \\ \text{II:} & a = 2r + 2s + 1 \quad (s = 1, 2, \dots) \\ \text{III:} & a = 2r - s \quad (s = 0, 1, 2, \dots), \end{array}$$

and solve one equation at a time from the sets I, II, III in turn.

I. Assume that $d_{2r-s+1}, d_{2r-s+2}, \dots, d_{2r+2s-1}$ are known. When $a = 2r + 2s$

the summation in (26) is from $2r-s+1$ to $2r+2s$. Thus all quantities are known except d_{2r+2s} . By Lemma 5, $b_{2r+2s, 2r+2s} = 1$ so that the coefficient of d_{2r+2s} on the left-hand side of (28) is 2 and d_{2r+2s} may be found.

II. Now $d_{2r-s+1}, \dots, d_{2r+2s}$ have been determined. The summation in (26) for $a = 2r+2s+1$ is from $2r-s$ to $2r+2s+1 = a$. All d 's that appear are known except d_a and d_{2r-s} . By Lemma 5, the coefficient of d_a is 0 while that of d_{2r-s} is $(-1)^{2r-s}$ so that d_{2r-s} may be expressed in terms of known quantities.

III. Finally assume $d_{2r-s}, \dots, d_{2r+2s}$ are known. If $a = 2r-s$ the summation in (26) is from $2r$ to $2r+2s+1$. Every d_j appearing has been evaluated except $d_{2r+2s+1}$ whose coefficient is $(-1)^a$ by Lemma 5. Thus $d_{2r+2s+1}$ may also be found.

The order in which we solve the equations (26) is this. First take $s=0$ in Case III, then $s=1$ successively in Cases I, II, III. Repeat with $s=2, 3, \dots$ until d_{3r} is determined. This occurs either in Case I with $r=2s$ or in Case III when $r=2s+1$. By then $d_{2r-s+1}, \dots, d_{3r}$ have been determined. Finally d_1, \dots, d_{2r-s} are found by using (26) with $a=1, \dots, 2r-s$ respectively, for these equations are simply

$$d_a + \sum_{j=2r}^{3r} b_{aj} d_j = g_a.$$

The proof that the equations (26) may be solved when $n=6r+3$ and $6r+5$ is similar and will not be given. When n is even, however, many relations of dependence occur among the equations (26) and it is impossible to solve for the d_j except in the following case.

THEOREM 8. *The values of $(2r, 2r, 2r)$ and $(2r-1, 2r, 2r+1)$ may be found explicitly as a polynomial in c_2, \dots, c_{6r} with rational coefficients.*

Proof. When $a=2r$ and $n=6r$, equation (26) becomes $d_{2r} + 2d_{2r} = g_{2r}$ by Lemmas 4 and 5. Hence $(2r, 2r, 2r) = g_{2r}/3$. Next by (4), $(2+1, 2r, 2r+1) = (2r, 2r, 2r)/2$.

If $n=6$ we may evaluate all $(r, s, n-r-s)$ according to our theorems except $(3, 1, 2)$, $(2, 0, 4)$ and $(4, 0, 2)$. These too may be found using (4) since $(3, 1, 2) = (3, 2, 1) - (2, 2, 2)$, $(2, 0, 4) = (1, 2, 3) - (1, 1, 4)$, and $(4, 0, 2) = (4, 1, 1) - (3, 1, 2)$. We remark in particular that $(2, 2, 2) = c_6/3$.

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ESSENTIAL AND NON ESSENTIAL FIXED POINTS.*

By M. K. FORT, JR.

1. Introduction. Let (X, d) be a compact metric space having the fixed point property. We denote by X^X the set of all continuous functions on X into X . If f and g are in X^X , we define

$$\rho(f, g) = \sup[d(f(x), g(x)) | x \in X].$$

It is well known that (X^X, ρ) is a complete metric space.

Let p be a fixed point of $f \in X^X$. We say that p is an *essential fixed point* of f if corresponding to each neighborhood U of p there is $\epsilon > 0$ such that g has a fixed point in U whenever $g \in X^X$ and $\rho(f, g) < \epsilon$.

The concept of essentiality for fixed points is a stability property which is quite analogous to the concept of stable value of a function. (See [1], p. 74). Several definitions of stable fixed points have been given in the past. However, no definition of stable fixed point with which the author is familiar is equivalent to the definition of essential fixed point which is given here. We will later prove that there exists a close connection between essential fixed points and essential mappings in the sense of homotopy.

2. An approximation theorem. Members of X^X , all of whose fixed points are essential, are especially well-behaved functions. We shall prove that each element of X^X can be approximated arbitrarily closely by such functions.

We shall make use of the hyperspace $(2^X, H)$ and of the theory of semi-continuous functions into 2^X . The points of 2^X are the non-empty compact subsets of X . If $A \in 2^X$ and $\epsilon > 0$, we denote by $U(\epsilon, A)$ the set of all points $x \in X$ for which there exists $y \in A$ such that $d(x, y) < \epsilon$. The set 2^X is metrized by H , where

$$H(A, B) = \inf[\epsilon | A \subset U(\epsilon, B) \text{ and } B \subset U(\epsilon, A)]$$

whenever A and B are in 2^X . It is well-known that $(2^X, H)$ is a compact metric space.

A function T on a topological space S into 2^X is upper semi-continuous [lower semi-continuous] at $p \in S$ if corresponding to each $\epsilon > 0$ there is a

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neighborhood V of p such that $T(x) \subset U(\epsilon, T(p))$ [$T(p) \subset U(\epsilon, T(x))$] for all $x \in V$. It is clear that T is continuous at p (with respect to the metric H) if and only if T is both upper and lower semi-continuous at p .

We next define a function F on X^X into 2^X by defining $F(f)$ to be the set of all fixed points of f whenever $f \in X^X$. The following lemma is due to Wehausen [2].

LEMMA 1. *The function F is upper semi-continuous.*

Proof. Let $f \in X^X$ and suppose $\epsilon > 0$. Define

$$\delta = \inf[d(x, f(x)) \mid x \in X - U(\epsilon, F(f))]$$

unless $X = U(\epsilon, F(f))$, in which case define $\delta = 1$. It is easily verified that $\delta > 0$, and that if $g \in X^X$ and $\rho(f, g) < \delta$ then $F(g) \subset U(\epsilon, F(f))$. This proves that F is upper semi-continuous.

LEMMA 2. *Each fixed point of $f \in X^X$ is essential if and only if f is a point of continuity of F .*

Proof. Assume that each fixed point of f is essential. Let ϵ be any positive number. For each $x \in F(f)$ there is a neighborhood $V(x)$ of f such that if $g \in V(x)$ then g has a fixed point in the $\epsilon/2$ -neighborhood of x . There exists a finite set x_1, \dots, x_n of points of $F(f)$ such that each point of $F(f)$ is within $\epsilon/2$ of some one of the x_k . If we now choose a neighborhood V of f which is contained in the intersection of $V(x_1), \dots, V(x_n)$ we see that if $g \in V$ then $F(g) \subset U(\epsilon, F(f))$. We have therefore proved that F is lower semi-continuous at f . This fact, in view of Lemma 1, proves that F is continuous at f .

We now assume that F is continuous at f . Let $p \in F(f)$ and let U be a neighborhood of p . Choose $\epsilon > 0$ such that the ϵ -neighborhood of p is contained in U . Now choose $\delta > 0$ such that if $\rho(f, g) < \delta$ for $g \in X^X$, then $H(F(f), F(g)) < \epsilon$. Thus if $g \in X^X$ and $\rho(f, g) < \delta$, then g has a fixed point in U . This proves that p is an essential fixed point of f .

THEOREM 1. *If $f \in X^X$ and $\epsilon > 0$, then there exists $g \in X^X$ such that $\rho(f, g) < \epsilon$ and such that every fixed point of g is essential.*

Proof. It is known, (see [3]), that the points of continuity of an upper semi-continuous set-valued function form a residual set. Since F is upper semi-continuous and X^X is a complete metric space, we see that F is continuous at each point of a set dense in X^X . Our theorem now follows at once from Lemma 2.

3. Existence theorems. It is easy to see that there may exist members of X^X which have no essential fixed point. A simple example is obtained by taking X to be a Euclidean n -cell. For this special case, every point of X is a fixed point of the identity transformation, but no point is an essential fixed point.

It is of interest to know conditions under which a transformation will have essential fixed points. We prove two theorems in this section which give sufficient conditions for the existence of an essential fixed point. The second of these theorems, Theorem 3, contains in the hypothesis a rather strong restriction on the space X . The author does not know whether or not the conclusion of this theorem is true if the hypothesis is weakened by allowing X to be an arbitrary compact metric space.

THEOREM 2. *If f has a single fixed point, then this point is an essential fixed point of f .*

Proof. Suppose $\epsilon > 0$. By the upper semi-continuity of F , there exists a neighborhood V of f such that if $g \in V$ then $F(g) \subset U(\epsilon, F(f))$. Since $F(f)$ contains but a single point, this implies that $F(f) \subset U(\epsilon, F(g))$. Thus if $g \in V$, then $H(F(f), F(g)) < \epsilon$. We have proved that F is continuous at f , and hence it follows from Lemma 2 that the fixed point of f is an essential fixed point.

THEOREM 3. *In addition to our standing hypotheses concerning X , we shall assume that X is a topological n -manifold, either with or without boundary. If $f \in X^X$ and the set $F(f)$ of fixed points of f is totally disconnected, then f has at least one essential fixed point.*

Proof. Let us assume that f does not have an essential fixed point. We shall show that this implies the existence of a function $g \in X^X$ which does not have a fixed point. Since X has the fixed point property, this is a contradiction.

Each point $x \in X$ has a neighborhood $N(x)$ whose closure is homeomorphic to a convex subset $C(x)$ of Euclidean n -space. For each $x \in F(f)$ choose a neighborhood $U(x)$ with the following properties: $U(x) \subset N(x)$, the intersection of $F(f)$ and the boundary of $U(x)$ is vacuous, there exist members of X^X arbitrarily close to f which do not have fixed points in $U(x)$. Let us cover $F(f)$ with a finite set $U(x_1), \dots, U(x_k)$ of these neighborhoods.

Now define $F_1 = F(f) \cap U(x_1)$ and define

$$F_i = F(f) \cap U(x_i) - \bigcup_{j=1}^{i-1} F_j$$

if $1 < i \leq k$. The sets F_i are closed, pairwise disjoint sets; and we may suppose the sets $U(x_1), \dots, U(x_k)$ chosen so that each set F_i is non-vacuous.

Choose open sets $V_1, \dots, V_k, W_1, \dots, W_k$ with the following properties:

$F_i \subset V_i \subset \bar{V}_i \subset W_i \subset \bar{W}_i \subset U(x_i)$, $\bar{W}_i \cap \bar{W}_j$ is the empty set if $i \neq j$,

$$f(\bar{W}_i) \subset N(x_i).$$

Let h_i be a homeomorphism of $\bar{N}(x_i)$ onto $C(x_i)$ for each i , $1 \leq i \leq k$.

Define $\epsilon = \inf[d(x, f(x)) | x \in X - \bigcup_{i=1}^k V_i]$. Choose $\delta > 0$ so that if $p, q \in C(x_i)$ and $\|p - q\| < \delta$, then $d(h_i^{-1}(p), h_i^{-1}(q)) < \epsilon$ for $1 \leq i \leq k$; choose $\eta > 0$ so that if $u, v \in \bar{N}(x_i)$ and $d(u, v) < \eta$, then $\|h_i(u) - h_i(v)\| < \delta$ for $1 \leq i \leq k$. Next, we may choose functions $g_i \in X^X$ for $1 \leq i \leq k$ such that $p(f, g_i) < \eta$, g_i does not have a fixed point in \bar{V}_i , and $g_i(\bar{W}_i) \subset N(x_i)$.

If $p \in W_i - \bar{V}_i$ we define $a_i(p) = \inf[d(p, x) | x \in \bar{V}_i]$ and $b_i(p) = \inf[d(p, x) | x \in X - W_i]$. We assume that each W_i has been chosen small enough so that $X - W_i$ is non-vacuous. Let $A_i(p) = a_i(p)/(a_i(p) + b_i(p))$ and $B_i(p) = b_i(p)/(a_i(p) + b_i(p))$. We now define $g(p)$ to be $f(p)$, if $p \in X - \bigcup_{i=1}^k W_i$; $g_i(p)$, if $p \in \bar{V}_i$ for some i , $1 \leq i \leq k$; $h_i^{-1}(A_i(p)h_i f(p) + B_i(p)h_i g_i(p))$, if $p \in W_i - \bar{V}_i$ for some i , $1 \leq i \leq k$. The function g belongs to X^X and it is easily seen that $d(p, g(p)) > 0$ for all $p \in X$. This is a contradiction.

4. Local study of essential fixed points. Although Theorem 3 is frequently useful in proving the existence of an essential fixed point of a function f , this theorem is of no value in deciding which fixed points are essential if f has more than one fixed point. For example, let X be the unit disk in the complex plane and let $f(z) = z^2$. The fixed points of f are 0 and 1. We know by Theorem 3 that at least one of these is essential, but we do not know which ones are essential and which ones are not essential. It is desirable to have theorems which we may use to prove that a fixed point is essential merely by examining the behavior of the function in a neighborhood of the point. The following is a simple theorem of this type.

THEOREM 4. *If $f \in X^X$, $p \in X$ and p has arbitrarily small neighborhoods V such that \bar{V} has the fixed property and $f(\bar{V}) \subset V$, then p is an essential fixed point of f .*

Proof. It is clear that p is a fixed point of f .

Let U be any neighborhood of p . Choose a neighborhood V of p such

that $\bar{V} \subset U$, \bar{V} has the fixed point property, $f(\bar{V}) \subset V$, and $X - V$ is non-vacuous. Let us define $\epsilon = \inf[d(x, y) \mid x \in f(\bar{V}), y \in X - V]$. Since $f(\bar{V})$ and $X - V$ are disjoint, non-vacuous, compact sets, we see that ϵ is a positive number.

Now let $g \in X^X$ and assume that $\rho(g, f) < \epsilon$. Then $g(\bar{V}) \subset V \subset \bar{V}$, and since \bar{V} has the fixed point property, g has a fixed point in \bar{V} and hence in U . This proves that p is an essential fixed point of g .

Using the above theorem, it is now easy to show that 0 is an essential fixed point of the function $f(z) = z^2$ for z in the unit complex disk.

It is very easy to describe situations where Theorem 4 is of no use. For example, consider the function $f(z) = z/|z|^{\frac{1}{2}}$ defined for all z in the unit complex disk, $f(0)$ being defined to be 0. The behavior of this function on neighborhoods of 0 is fairly typical of the situation we may expect in general, in that the function f does not transform small neighborhoods of 0 into themselves. This suggests that one might prove stronger theorems than Theorem 4 if one had as tools fixed point theorems for mappings of sets into subsets of some containing space rather than merely into subsets of themselves. We prove one such theorem in the next section.

5. A fixed point theorem. Let n be a positive integer and let E_n denote Euclidean n -space. The set of all points x in E_n such that $\|x\| \leq r$ will be denoted by $K(r)$, and the set of all points x in E_n such that $\|x\| = r$ will be denoted by $S(r)$. If $x \in E_n$ and x is not the origin, we let $P(x) = x/\|x\|$.

Throughout this section we shall assume that r is a positive number and that f is a continuous function on $K(r)$ into E_n such that $f(x) \neq x$ for all $x \in S(r)$. We associate with f a mapping f^* of $S(1)$ into $S(1)$ by defining $f^*(x) = P(rx - f(rx))$ for all $x \in S(1)$.

THEOREM 5. *If f^* is an essential mapping (not homotopic to a constant mapping) then f has a fixed point.*

Proof. Suppose that f does not have a fixed point. In this case $trx \neq f(trx)$ for all $x \in S(1)$ and $0 \leq t \leq 1$. Thus we may define $D(x, t) = P(trx - f(trx))$ for all $x \in S(1)$ and $0 \leq t \leq 1$. The function D is clearly a homotopy of f^* to a constant function.

Example 1. Let f be such a mapping that $f(K(r)) \subset K(r)$. It is easily seen that f^* is homotopic to the identity mapping if one considers the homotopy defined by $D(x, t) = P(rx - tf(rx))$. The Brouwer fixed point

theorem now follows from the fact that the identity mapping on $S(1)$ is essential.

Example 2. Let $n=2$ and let g be the mapping of $K(1)$ into E_2 which takes the point (u, v) into $(2u, v/2)$. The homotopy defined by $D((u, v), t) = P((tu - 2u, 2v - 3tv/2))$ deforms g^* into the mapping which takes (u, v) into $(-u, v)$. This mapping, being a square root of the identity, is essential. Thus g^* is essential.

Now there exists $\epsilon > 0$ such that $\|x - g(x)\| \geq \epsilon$ for all $x \in S(1)$. It is easy to see that if f is any mapping of $K(1)$ such that $\|f(x) - g(x)\| < \epsilon$ for all $x \in S(1)$, then f^* is homotopic to g^* and consequently is essential. Any such mapping f , by Theorem 5, has a fixed point. Notice that f is required to be close to g only on $S(1)$ and not on all of $K(1)$.

THEOREM 6. *If f^* is homotopic to a constant, then there exists a mapping g on $K(r)$ into E_n without fixed points such that $g(rx) = f(rx)$ for all $x \in S(1)$. Moreover, if $\delta > 0$, we may choose g so that $\rho(g, I) \leq \rho(f, I) + \delta$, where I is the identity mapping on $K(r)$.*

Proof. Let D be such a homotopy that $D(x, 1) = f^*(x)$ and $D(x, 0)$ is constant. Define $g(trx) = trx - [t\|rx - f(rx)\| + (1-t)\delta]D(x, t)$ for all $x \in S(1)$ and $0 \leq t \leq 1$. It is easily seen that g is a continuous function on $K(r)$ into E_n which has no fixed points. Also,

$$g(rx) = rx - \|rx - f(rx)\| D(x, 1) = rx - \|rx - f(rx)\| f^*(x) = f(rx)$$

for all $x \in S(1)$.

Moreover,

$$\begin{aligned} \|g(trx) - trx\| &= [t\|rx - f(rx)\| + (1-t)\delta]\|D(x, t)\| \\ &\leq \|rx - f(rx)\| + \delta \leq \rho(f, I) + \delta \end{aligned}$$

for all $x \in S(1)$ and $0 \leq t \leq 1$. Thus $\rho(g, I) \leq \rho(f, I) + \delta$.

6. Isolated fixed points on manifolds. Let X be a compact n -manifold (either with or without boundary) which has the fixed point property. Let $f \in X^X$ and let p be an isolated fixed point of f . In case X has a non-vacuous boundary, we shall also assume that p is not a boundary point of X . Let N be a neighborhood of p whose closure is homeomorphic to the closed unit sphere $K(1)$ in E_n . We may assume that h is a homeomorphism of \bar{N} onto $K(1)$ which takes p into the origin of E_n . We also assume that N has been chosen small enough so that the only fixed point of f in \bar{N} is p .

For all sufficiently small positive numbers r , say $0 < r \leq R$, $fh^{-1}(K(r)) \subset N$ and hence $fh^{-1}|K(r)$ is defined and is a function which we shall denote by f_r . Since the only fixed point of f_r is the origin, we may associate with each f_r , $0 < r \leq R$, a mapping f_r^* of $S(1)$ into $S(1)$ by defining $f_r^*(x) = P(rx - f_r(rx))$ for each $x \in S(1)$. It is clear that f_r^* is homotopic to f_R^* if $0 < r \leq R$.

THEOREM 7. *The point p is an essential fixed point of f if and only if f_R^* is an essential mapping.*

Proof. Let us assume that f_R^* is an essential mapping. Let U be any neighborhood of p . Choose r , $0 < r < R$, such that $h^{-1}(K(r))$ and $fh^{-1}(K(r))$ are both contained in U . Now choose $\epsilon > 0$ such that if $\rho(f, g) < \epsilon$ then $gh^{-1}(K(r))$ is contained in both U and N . For all g such that $\rho(f, g) < \epsilon$, we may define g_r and g_r^* in the obvious manner. We may assume that ϵ has been chosen small enough so that g_r^* and f_r^* are homotopic. Thus, if $\rho(g, f) < \epsilon$, g_r^* is essential and it follows from Theorem 5 that g_r has a fixed point q in $K(r)$. The point $h^{-1}(q)$ is in U and is a fixed point of g . This proves that p is an essential fixed point of f .

Now let us assume that f_R^* is non-essential. Let ϵ be any positive number. We will show that there exists $g \in X^X$ such that $\rho(f, g) < \epsilon$ and such that g does not have a fixed point in N . Choose r , $0 < r < R$, such that $h^{-1}(K(r))$ and $fh^{-1}(K(r))$ are both contained in the $\epsilon/4$ neighborhood of p . Now choose s , $0 < s < r$, such that $\rho(f_s, I) + 2s < r$. By Theorem 6 there exists a function g_s on $K(s)$ into E_n such that $g_s(x) = f_s(x)$ for all $x \in S(s)$ and such that g_s has no fixed points. Moreover, we may assume that $\rho(g_s, I) + s \leq \rho(f_s, I) + 2s < r$, and hence $g_s(K(s)) \subset K(r)$. We now define $g(x) = f(x)$ if $x \in X - h^{-1}(K(s))$ and $g(x) = h^{-1}g_sh(x)$ if $x \in h^{-1}(K(s))$. It is easy to verify that g has the required properties. We have thus shown that p is not an essential fixed point of f .

Example 3. Let X be the set of all points (u, v) in E_2 such that $\max(|u|, |v|) \leq 1$. Let f be the function on X which transforms the point $(u^{1/3}, v^3)$. Locally, f behaves similarly to the mapping defined in Example 2. Combining the methods used in Example 2 with Theorem 7, it is easy to prove that the origin is an essential fixed point of f .

7. A fixed point theorem for product spaces. Let X be a compact metric space having the fixed point property, and let T be the closed unit interval of real numbers. We denote by f a function on $X \times T$ into $X \times T$, and denote the transform of the point (x, t) under f by $(f_t(x), g(x, t))$.

THEOREM 8. *If for each t , $0 \leq t \leq 1$, the function f_t of X into X has only essential fixed points, then f has a fixed point.*

Proof. For each t let $Z(t) = F(f_t)$. Using Lemma 2 it is easy to prove that Z is a continuous function on T into 2^X . It follows that there exists a connected set C contained in $X \times T$ such that if $(x, t) \in C$ then $x \in Z(t)$, and such that there exist $x_0, x_1 \in X$ for which $(x_0, 0) \in C$ and $(x_1, 1) \in C$. Consider the continuous real valued function on C defined by $h(x, t) = g(x, t) - t$. Since $h(x_0, 0) \geq 0$ and $h(x_1, 1) \leq 0$, it follows that $h(x^*, t^*) = 0$ for some $(x^*, t^*) \in C$. Thus $g(x^*, t^*) = t^*$ and $f_{t^*}(x^*) = x^*$, and we see that (x^*, t^*) is a fixed point of f .

The hypothesis of the above theorem is quite restrictive. It is not known whether or not there exist spaces X and Y with the fixed point property for which $X \times Y$ does not have the fixed point property. Consequently, one would hope for a much stronger theorem than Theorem 8.

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ON THE COMMUTATOR GROUP OF A SIMPLE ALGEBRA.*¹

By SHIANGHAW WANG.

Introduction. Let A be an algebra. If α in A is a commutator: $\alpha = \beta\gamma\beta^{-1}\gamma^{-1}$, clearly, its norm (in any sense) must be unity. It follows that any element in the commutator group must be of norm 1.

Now, consider as an example the algebra D of quaternions over the field R of real numbers. Any element α in D is contained in a quadratic extension of R , which is by necessity a field $R(i)$, isomorphic to the field of complex numbers. If α is of reduced norm 1: $\alpha\bar{\alpha} = 1$, it can be written in the form $\cos \theta + i \sin \theta = e^{\theta i}$. Since there exists an element j in D such that $j\xi j^{-1} = \bar{\xi}$ for any ξ in $R(i)$, we have $e^{\theta i} = e^{\frac{1}{2}\theta i} j (e^{\frac{1}{2}\theta i})^{-1} j^{-1}$. This shows any element of reduced norm 1 in D is a commutator.

Nakayama and Matsushima² proved that any element of reduced norm 1 in a p -adic division algebra is a product of at most three commutators. Combining this result with the above simple fact in the quaternion algebra, we may say that any element of reduced norm 1 in a "local" division algebra is in the commutator group. It is but a step to infer the corresponding statement for any "local" simple algebra.

Our main purpose here is to prove that the statement holds "in the large," i. e., the commutator group of the group of all regular elements in a simple algebra A over an algebraic number field k coincides with the group of all elements of reduced norm 1. From this theorem and the norm theorem of Hasse-Schilling-Maass, we infer that the factor commutator group of A is isomorphic in a natural way to the group of all elements in k , which are positive at all infinite primes where A is ramified.

Whether the theorem is true for any simple algebra we cannot yet decide. Nevertheless, we shall prove its general validity in the special case where the index of the algebra is square free.

The commutator group of a simple ring which is not a pure division ring is a generalization of the classical special linear group. Apart from

* Received February 21, 1949.

¹ This investigation of the multiplicative group of a simple algebra was suggested to the author by Professor E. Artin.

² T. Nakayama and Y. Matsushima, "Über die multiplikative Gruppe einer p -adischen Divisionsalgebra," *Proceedings of the Imperial Academy of Japan*, 19, 1943.

certain special cases to be discussed in I below, the question of its structure was settled by Dieudonné.³ But the group of a pure division ring is entirely different. The structure problem in this case seems very difficult even when a division algebra over an algebraic number field is concerned.

I. Commutator Groups of Simple Rings.

Let A be a simple ring, i. e., a full matrix ring over a division ring D . The set of all regular elements in A forms a group which will be denoted by the same symbol denoting the ring itself.

Suppose A is of degree $m > 1$ over D : $A = M_{(m)} \times D$, and let E_{ij} , $i, j = 1, \dots, m$, form the usual matrix basis. E being the unit matrix, let $B_{ij}(\lambda) = E + \lambda E_{ij}$, $i \neq j$, and let $E(\alpha) = E + (\alpha - 1)E_{mm}$. Then the set of all $B_{ij}(\lambda)$ generates the commutator group A' of A and any coset of A' in A can be represented by a matrix $E(\alpha)$. The central elements of A or of A' are contained in the center k of D .

Dieudonné proved in [2] that the factor commutator group A/A' is isomorphic to D/D' and the isomorphism can be brought up by the function Δ which maps the coset of $E(\alpha)$ to the coset of α in D/D' . $\Delta(X)$ was called the determinant of the matrix X . The following lemma is an immediate consequence of this result of Dieudonné:

LEMMA 1. *The matrix $\alpha_1 E_1 + \dots + \alpha_m E_{mm}$ is in A' if and only if the product $\alpha_1 \dots \alpha_m$ is in D' .*

We shall call a group simple if any proper normal subgroup is contained in the center. Dieudonné determined the structure of A' by showing that it is simple when $m > 2$, or when $m = 2$ and the center k of D is not the prime field R_2 , R_3 , or R_5 of characteristic 2, 3, or 5 respectively. We wish to show that this restriction can be removed. In fact, we shall prove that the group A' is simple, when $m > 2$, or when $m = 2$ and $D \neq R_2$ or R_3 .

Proof. The general proof given below does not apply when $D = R_5$. For this singular case, see Dickson [3], p. 85.⁴

Let $A = M_{(2)} \times D$, $D \neq R_2, R_3, R_5$. Let N be a normal subgroup of A , containing a matrix

$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \notin A' \cap k.$$

³ J. Dieudonné, "Les déterminants sur un corps non commutatif," *Bulletin de la Société Mathématique de France*, vol. 71 (1943).

⁴ L. E. Dickson, *Linear Groups*, Leipzig, 1901.

Following the method in [2], we shall first construct a matrix of the form $B_{12}(\lambda_0)$ from X and its transforms.

Suppose one of β and γ is not 0. We may assume $\beta \neq 0$. For, if $\gamma \neq 0$, X may be replaced by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \delta - \gamma & \\ -\beta & \alpha \end{pmatrix}.$$

Furthermore, we may assume $\alpha = 0$, because

$$(1) \quad \begin{pmatrix} 1 & 0 \\ -\beta^{-1}\alpha & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\beta^{-1}\alpha & 1 \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ \gamma_1 & \delta_1 \end{pmatrix}.$$

For $\xi \in D$, k_ξ will denote a subfield of D , containing the field $k(\xi)$, and containing at least one element outside k in case $k = R_2, R_3$ or R_5 .

If $\eta \in k_\gamma$, we have

$$(2) \quad \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}^{-1} \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} = \begin{pmatrix} \eta^2 & * \\ 0 & \beta^{-1}\eta^{-1}\beta\eta^{-1} \end{pmatrix} = Y.$$

When $k \neq R_2, R_3, R_5$, we choose η in k such that $\eta^4 \neq 1$. When $k = R_2, R_3$ or R_5 , we choose η in k_γ outside k such that $\eta^2 \notin k$. In both cases, Y has the property that either its two diagonal elements are distinct or they do not belong to k .

If $\beta = \gamma = 0$ in the original matrix X , X has already the above property, because it is not in the center of A' by assumption. Hence, N always

contains a matrix $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ where $\alpha \neq \delta$ or $\alpha = \delta$ but $\alpha \notin k$. So, there exists an element $\xi \in D$ for which $\alpha\xi \neq \xi\delta$. We have

$$(3) \quad \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \xi - \alpha\xi\delta^{-1} \\ 0 & 1 \end{pmatrix} = B_{12}(\lambda_0),$$

where $\lambda_0 = \xi - \alpha\xi\delta^{-1} \neq 0$.

Now, consider a field k_{λ_0} . Starting with $B_{12}(\lambda_0)$ as the original matrix X , we first obtain a matrix of the form $\begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$ as in (1), $\beta, \gamma, \delta \in k_{\lambda_0}$. This time, we choose $\eta \in k_{\lambda_0}$ such that $\eta^4 \neq 1$ and obtain a matrix of the form $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ as in (2), $\alpha, \beta, \delta \in k_{\lambda_0}$, $\alpha \neq \delta$. Taking $\xi = (1 - \alpha\delta^{-1})^{-1}$ in (3), we obtain the matrix $B_{12}(1)$.

Let $\lambda \in D$ be arbitrary. Starting with $B_{12}(1)$ as our matrix X , we repeat the above process, keeping the coefficients of the matrices inside a field k_λ . Setting $\xi = \lambda(1 - \alpha\delta^{-1})^{-1}$ in (3), we obtain $B_{12}(\lambda)$. Since

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1},$$

we have finally the matrix $B_{21}(\lambda)$. So, $N = A'$ and the statement is proved.

Since, as we shall prove, the commutator group of a simple algebra over an algebraic number field coincides with the group of all elements of norm 1, it is conceivable that the group D/D' might always be isomorphic to a subgroup of the center k and Dieudonné's determinant of a matrix could then be realized by an element in k . As the following example shows, this is not the case:

Let K be the field of all power series over R_2 of the form

$$\sum_{\nu=r}^{\infty} a_{\nu} x^{\nu}, \quad a_{\nu} = 0 \text{ or } 1, \quad r \geq 0.$$

We introduce a second variable y and consider all the double power series

$$\sum_{\mu=s}^{\infty} \alpha_{\mu} y^{\mu}, \quad \alpha_{\mu} \in K, \quad s \geq 0.$$

We add and multiply them formally applying the rule $y\alpha = \alpha^{\sigma}y$, $\alpha \in K$, where σ is the automorphism of K defined by the equation

$$(4) \quad x^{\sigma} = x + x^2.$$

As can be readily verified, these double power series form a division ring D . Now, in a power series field, an automorphism of infinite period, which can be defined by an equation like (4), leaves fixed only the coefficient field. Since σ is obviously of infinite period, it can be easily seen from the above remark that the center of D is exactly $R_2 = \{0, 1\}$.

Let $\xi = \sum_{\mu=s}^{\infty} \alpha_{\mu} y^{\mu}$ be in D , $\alpha_s \neq 0$. We define $|\xi| = y^s$. By the rule of multiplication, we have $|\xi\eta| = |\xi| |\eta|$. So, this is a homomorphism of D onto the infinite cyclic group generated by y . Since the latter is abelian, D/D' is certainly not trivial.

II. Proof of the Main Theorem.

Let A be a central simple algebra over k . By the norm $N\alpha$ of $\alpha \in A$ we mean the reduced norm $N_{Ak}(\alpha)$ of α over k . If α is in the maximal subfield K of A , then $N\alpha = N_{Kk}(\alpha)$.

Let K be a cyclic maximal subfield over k and let σ be a generating

automorphism. Then there exists an element τ in A such that $\tau\xi\tau^{-1} = \xi^\sigma$ for any $\xi \in K$. If $N\alpha = 1$, $\alpha \in K$, there exists an element ξ in K such that $\alpha = \xi^{\sigma^{-1}}$ by a theorem of Hilbert. So, $\alpha = \tau\xi\tau^{-1}\xi^{-1}$ and we have proved

LEMMA 2. Any element of norm 1 in A , that is contained in a cyclic maximal subfield, is a commutator.

Let D be a central division algebra over k and F a field of degree m over k .

LEMMA 3. If $\alpha \in D$ is in the commutator group D'_F of D_F , then α^m is in D' .

Proof. Consider the algebra $A = M_{(m)} \times D$. Since F is isomorphic to maximal subfield of $M_{(m)}$, we may regard $D_F = F \times D$ as a subalgebra of A . Hence, $\alpha \in D'_F \subset A'$. By Lemma 1, $\alpha^m \in D'$.

Now, let K be a splitting field of $D = D_{(n)}$, of degree n over k . If $\alpha \in D$ has norm 1 in D , it still has norm 1 in $D_K = M_{(n)}$. But an element of norm 1 in a full matrix algebra is nothing but a matrix of determinant 1. So, $\alpha \in D'_K$. This, together with Lemma 3, proves

LEMMA 4. If $N\alpha = 1$ in $D = D_{(n)}$, then $\alpha^n \in D'$.

Let k be a p -adic field and $D = D_{(n)}$ be a central division algebra over k . Then D contains an unramified maximal subfield W . For a suitably chosen generating automorphism σ of W and a suitably chosen prime element π in D , we have

$$D = W + W\pi + \cdots + W\pi^{n-1}; \pi\xi\pi^{-1} = \xi^\sigma \text{ for } \xi \in W, \pi^n = p.$$

So, D admits an irreducible representation over W , in which the element $\alpha = \alpha_0 + \alpha_1\pi + \alpha_2\pi^2 + \cdots + \alpha_{n-1}\pi^{n-1}$, $\alpha_i \in W$, is represented by the matrix

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{n-1} \\ p\alpha_{n-1}^\sigma & \alpha_0^\sigma & \cdots & \alpha_{n-2}^\sigma \\ p\alpha_{n-1}^{\sigma^2} & p\alpha_{n-2}^{\sigma^2} & \cdots & \alpha_0^{\sigma^{n-1}} \end{pmatrix}.$$

It can be shown that α is integral if and only if so are $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$.

Now, let α be a unit in D . Let q be the number of residue-classes of p in k . We have $\alpha_0 \equiv \omega \pmod{p}$, ω being a $(q^n - 1)$ -th root of unity in W . It follows from the above representation that $N\alpha \equiv N\alpha_0 \equiv N\omega \pmod{p}$.

The following lemma is a slight improvement of a result of Nakayama and Matsushima, referred to in the introduction:

LEMMA 5. Any element α of norm 1 in D is a product of at most two commutators.

Proof. Since α is of norm 1, it is a unit. Using the notations introduced above, we have $\alpha \equiv \omega \pmod{\pi}$ and $1 = N\alpha \equiv N\omega \pmod{p}$. Since $N\omega$ is a $(q^n - 1)$ -th root of unity, the latter congruence implies $N\omega = 1$. So, $\omega^{1+q+\dots+q^{n-1}} = 1$, or ω is a $(q^n - 1)/(q - 1)$ -th root of unity.

Suppose ω is a primitive $(q^n - 1)/(q - 1)$ -th root of unity. If $v < n$, we have $q^v - 1 < (q^n - 1)/(q - 1)$. So, ω is not a $(q^v - 1)$ -th root of unity for $v < n$, and therefore generates the residue-class field of π in D (i. e., the residue-class field of p in k by adjoining the class of ω). It follows that α also generates the residue-class field. Consequently, α generates a maximal unramified subfield of D . Since α is in a cyclic maximal subfield, it is a commutator.

If ω is not primitive, let $\omega_1 \in W$ be a primitive $(q^n - 1)/(q - 1)$ -th root of unity. Then $\omega_1^{-1}\alpha \equiv \omega_1^{-1}\omega \pmod{\pi}$, and $\omega_1^{-1}\omega$ is primitive. Since both ω_1 and $\omega_1^{-1}\alpha$ are commutators, α is in this case a product of two commutators.

Observe that, if we have proved that any element of norm 1 in D is in D' , D being an arbitrary division algebra, we may conclude immediately that any element of norm 1 in $A = M_{(m)} \times D$ is in A' . For, any element $\alpha \in A$ is equivalent to a diagonal matrix $E(\beta) \pmod{A'}$. If $N\alpha = 1$, we have $N_{Dk}(\beta) = 1$ and $\beta \in D'$. Consequently, $E(\beta) \in A'$ and $\alpha \in A'$.

Let $A = A_{(n)}$ be a central simple algebra over the p -adic field k , and $\omega_1, \omega_2, \dots, \omega_{n^2}$ form a basis. Let $\xi = t_1\omega_1 + \dots + t_{n^2}\omega_{n^2}$ be a general element of A . We valueate A by defining $|\xi| = \max(|t_1|, \dots, |t_{n^2}|)$. This is a valuation of A in a modified sense, inasmuch as the condition $|\xi\eta| = |\xi||\eta|$ for an orthodox valuation is not necessarily satisfied. Nevertheless, it turns A into a complete normed vector space over k , and different bases give rise to topological equivalent spaces. Needless to say, any subalgebra is a linear subspace and therefore, as a point set, is closed in A . Furthermore, if the subalgebra is a field, the valuation introduced above induces in it the same topology as its standard valuation does.

As in any normed vector space, $\xi + \eta$, $\xi - \eta$, $a\xi$ are of course continuous functions of ξ, η . Since the coefficients of $\xi\eta$ are polynomials in those of ξ, η , $\xi\eta$ is also a continuous function of ξ, η . Let $f_\xi(x) = x^n + a_1(t)x^{n-1} + \dots + a_n(t)$ be the principal polynomial of ξ . Then $a_1(t), \dots, a_n(t)$, as polynomials of t_1, \dots, t_{n^2} , are continuous functions of ξ . If ξ is regular, then $a_n(t) \neq 0$ and ξ^{-1} may be written in the form $\xi^{-1} = -a_n(t)^{-1}(\xi^{n-1} + a_1(t)\xi^{n-2} + \dots + a_{n-1}(t))$. Consequently, ξ^{-1} is a continuous function of ξ .

LEMMA 6. If α generates a maximal subfield K of A , then all elements in a sufficiently small neighborhood of α generate maximal subfields which are isomorphic to K .

Proof. Since there are only finitely many intermediate fields between K and k , and they are closed in K , there exists a neighborhood U_K of α in K , which does not intersect them. Since $f_\alpha(x)$ is irreducible over k , it has no multiple roots. So, there exists a neighborhood U_A of α in A such that, if $\xi \in U_A$, $f_\xi(x)$ will be sufficiently near $f_\alpha(x)$ so that it will have a root θ in U_K on account of Hensel's Lemma. Since θ does not belong to the intermediate fields, it generates K : $K = k(\theta)$. But then $f_\xi(x)$ must be irreducible. Consequently, $k(\xi)$ is a maximal subfield and is isomorphic to $k(\theta) = K$.

Let A be a central simple algebra over the algebraic number field k . Then $A^p = A \times k_p$ is the p -adic completion of A , p being a prime in k . If $\alpha_i \in A^{p_i}$, $i = 1, \dots, t$, are given, we can approximate them simultaneously by an element ξ in A . In fact, we need only approximate the coefficients.

LEMMA 7. Let $\epsilon > 0$ be given and let $\alpha_i \in A^{p_i}$, $i = 1, \dots, t$, be elements of norm 1. There exists an element $\xi \in A'$ such that $|\xi - \alpha_1|^{p_1} < \epsilon, \dots, |\xi - \alpha_t|^{p_t} < \epsilon$.

Proof. Since α_i is of norm 1, it is in $(A^{p_i})'$. Let

$$\alpha_i = \beta_{i1}\gamma_{i1}\beta_{i1}^{-1}\gamma_{i1}^{-1} \cdots \beta_{it_i}\gamma_{it_i}\beta_{it_i}^{-1}\gamma_{it_i}^{-1}, \quad \beta_{ij}, \gamma_{ij} \in A^{p_i}.$$

Let $\eta_{ij}, \xi_{ij} \in A$ be good approximations to β_{ij}, γ_{ij} at p_i and to 1 at $p_{i'}$, $i' = 1, \dots, t$, $i' \neq i$. Then

$$\xi = \prod_{j=1}^{s_1} (\eta_{1j}\xi_{1j}\eta_{1j}^{-1}\xi_{1j}^{-1}) \cdots \prod_{j=1}^{s_t} (\eta_{tj}\xi_{tj}\eta_{tj}^{-1}\xi_{tj}^{-1})$$

is a good approximation to α_i at p_i , $i = 1, \dots, t$.

THEOREM. Any element of norm 1 in a simple algebra over an algebraic number field is in the commutator group.

Proof. Let k be the center of the algebra. We prove the theorem in three steps, n being the index of the algebra:

i) $n = 1$, a prime. In this case, the assumption that k is an algebraic number field will not be used. By a remark made before, we may prove the

theorem only for a division algebra $D_{(l)}$. Let $\alpha_0 \in D$ be an element of norm 1. α_0 is contained in a maximal subfield K of D . If D is of characteristic l and K is inseparable over k , α_0 satisfies the equation $\alpha_0^l - 1 = 0$ which implies $\alpha_0 = 1$. So, we may assume K is separable over k . Let E be an extension of K , which is normal over k with the Galois group $G(k)$. Let $G(F)$ be an l -Sylow group of $G(k)$ with the fixed field F . Then $(F:k) = m$ is not divisible by l and D_F remains a division algebra. Consider the maximal subfield $K_F = K \times F$ of D_F and let the group of E over K_F be $G(K_F)$. Since $G(K_F)$ is a maximal subgroup in the p -group $G(F)$, it is normal in the same. Hence, K_F is normal and cyclic over F . $\alpha_0 \in K_F$ is therefore in D'_F by Lemma 2. Applying Lemma 3, we have $\alpha_0^m \in D'$. But Lemma 4 gives $\alpha_0^l \in D'$. Since $(l, m) = 1$, $\alpha_0 \in D$.

ii) $n = l^s$, $s > 1$. We prove the theorem by induction. Accordingly, assume the theorem has been proved for $n = l^{s-1}$. Let α_0 be an element in $D = D_{(l^s)}$ of norm 1.

Let S_0 be a finite set of finite primes in k , which contains

- (a) all finite primes where D is ramified,
- (b) all even primes,⁵
- (c) a prime which splits completely in the l -component of $k(\xi_{s+1})$ over k , ξ_{s+1} denoting a primitive l^{s+1} -th root of unity.

For $p \in S_0$, let K^p be a maximal unramified subfield of D^p , and let θ_p be an element of norm 1, which generates $K^p: K = k_p(\theta_p)$. For example, θ_p may be taken as a primitive $(q^{l^s} - 1)/(q - 1)$ -th root of unity in K^p , q being the number of residue-classes of p in k . Let $\xi \in D'$ be a sufficiently good approximation to the elements θ_p for all $p \in S_0$. Then $\alpha = \xi \alpha_0$ will be a sufficiently good approximation to the elements θ_p , and will generate on account of Lemma 6 maximal subfields in D^p isomorphic to K^p for all $p \in S_0$. It follows, first of all, that $K = k(\alpha)$ is a maximal subfield of D .

Now, let E , F , $G(k)$, $G(F)$, $G(K_F)$ have the same significance as in i). Passing a composition series of $G(F)$ through $G(K_F)$, we see K_F has a subfield Ω_1 , which is cyclic of degree l over F . We are now in the division algebra D_F . Let $v_1 = N_{K_F \Omega_1}(\alpha)$.

We shall denote the extension of S_0 in F by S_0 again. Let S^* be the

⁵ S. Wang, "On Grunwald's theorem," to appear in the *Annals of Mathematics*.

set of all primes in F , which are not relatively prime to ν_1 in Ω_1 . Since $N_{\Omega_1 F}(\nu_1) = 1$ and Ω_1 is unramified of degree l at the primes in S_0 , $S^* \cap S = 0$. Finally, let $q \notin S_0 + S^*$ be a prime which remains prime in Ω_1 and in the l -component of $F(\xi_s)$ over F .

By (c), Ω_1 is independent with $F(\xi_{s+1})$ over F . Applying Corollary 2 in [4] with $\tilde{C} = \Omega_1$, we construct a cyclic extension Γ of degree l^{s+1} over F such that

$$(A) \quad (\Gamma^v : F_p) = l^{s+1} \text{ for } p \in S_0,$$

$$(B) \quad \Gamma^v = F_p \text{ for } p \in S^* + q,$$

$$(C) \quad \text{if } \Gamma \text{ is ramified at any } p \text{ outside } S + S^* + q, p \text{ remains prime in } \Omega_1.$$

Let D_1 be the centralizer of Ω_1 in D_F . It is a division algebra of degree l^{s-1} over Ω_1 , and is isomorphic to the division algebra in $D_F \times \Omega_1$. So, if p_∞ is a real prime in F , where D_F is ramified, then D_1 is ramified over Ω_1 at $\mathfrak{P}_\infty | p_\infty$, if and only if \mathfrak{P}_∞ is real, i. e., if and only if Ω_1 is unramified over F at p_∞ . Let S_∞ be the set of all real primes in F , where D_F is ramified but where Ω_1 is unramified.

Let Ω^* be the subfield of Γ of degree l over F . Let \bar{S} be the set of all primes in F outside $S_0 + S^* + q$, where Ω^* is ramified over F . Put $\bar{S} = S_0 + S^* + \bar{S} + q + S_\infty$.

Let Γ_1 be the subfield of Γ of degree l^s over F . For $p \in S$, we define Γ^{*v} and $(\alpha_p, \Gamma^{*v}/F_p)$ as follows:⁶

$$(a_1) \text{ for } p \in S_0 + S^* + \bar{S} + q, \Gamma^{*v} = \Gamma_1^v \text{ and}$$

$$(\alpha_p, \Gamma^{*v}/F_p) = (\alpha_p, \Gamma_1^v/F_p),$$

(b₁) for $p \in S_\infty$, Γ^{*v} is complex and $(\alpha_p, \Gamma^{*v}/F_p)$ is defined in the only way possible.

Write $S' = S_0 + S^* + \bar{S} + q$. When $l=2$ and when $F(\xi_{s+1})$ is not cyclic over F , we have, by Lemma 3 in [4],

$$(5) \quad \prod_{p \in S'} ((\sec 2\pi/2^{t+1})^{2^{s-1}}, \Gamma_1^v/F_p) = 1.$$

⁶ This is the upper part of the orthodox norm residue symbol. The lower part, consisting of a horizontal line and the prime p , is omitted for printing convenience.

Since $(\sec 2\pi/2^{t+1})^{2^{s-1}}$ is totally positive, (5) implies that

$$\prod_{p \in S} ((\sec 2\pi/2^{t+1})^{2^{s-1}}, \Gamma^{*p}/F_p) = 1$$

and that condition (d) in the theorem in [4] is satisfied. Applying that theorem with $\tilde{C} = \Omega_1$, $C = \Omega$, we construct a cyclic extension Γ^* of degree l^s over F such that

$$(A_1) \quad \Omega^* \subset \Gamma^*,$$

$$(B_1) \quad \text{the } p\text{-adic completion of } \Gamma^* \text{ is } \Gamma^{*p} \text{ for } p \in S,$$

$$(C_1) \quad \text{if } \Gamma^* \text{ is ramified at any } p \text{ outside } S, p \text{ remains prime in } \Omega_1.$$

Consider the field $K_1 = \Omega_1 \times \Gamma^*$. This is a direct product, because q remains prime in Ω_1 , but splits completely in Γ^* by (B) and (a₁). We contend that v_1 is a norm everywhere in K_1 . In fact, at all primes extending those in S_∞ , $v_1 = N_{D_1\Omega_1}(\alpha)$ is a norm in D_1 and is therefore positive. At all primes extending those in S^* , K_1 splits completely over Ω_1 . Since v_1 is a unit at all finite primes not in S^* , it is a norm everywhere in K_1 except possibly at the finite primes \mathfrak{P} where K_1 is ramified over Ω_1 . Let p be the prime in F divisible by \mathfrak{P} . Then $p \in S_0 + \bar{S} +$ the primes in (C_1) . By the construction of K , Γ , and Γ^* , p remains prime in Ω_1 . We have

$$(v_1, K_1^{\mathfrak{P}}/\Omega_1^{\mathfrak{P}}) = (N_{\Omega_1^{\mathfrak{P}}} F_p(v_1), \Gamma^{*p}/F_p) = (1, \Gamma^{*p}/F_p) = 1.$$

Consequently, v_1 is a norm everywhere in K_1 , and there exists an element $\beta_1 \in K$ such that $N_{K_1\Omega_1}(\beta_1) = v_1$.

Consider the algebra $A = M_{(l)} \times D_F$ over F . It contains the algebra $A_1 = M_{(l)} \times D_1$ over Ω_1 . Since K_1 clearly splits D_1 and is of degree l^s over Ω_1 , it is a maximal subfield of A_1 . Consider the element

$$\gamma_1 = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & \alpha \end{pmatrix}$$

in A_1 . We have

$$N_{A_1\Omega_1}(\gamma_1) = N_{D_1\Omega_1}(\alpha) = v_1 = N_{K_1\Omega_1}(\beta_1) = N_{A_1\Omega_1}(\beta_1),$$

i. e., $N_{A_1\Omega_1}(\gamma_1) = N_{A_1\Omega_1}(\beta_1)$. By the hypothesis of induction, γ_1 differs from β_1 only by an element in $A'_1 \subset A'$.

Let us assume for a moment that it had already been proved that $\beta_1 \in A'$.

We would infer then $\gamma_1 \in A'$. Lemma 1 would give $\alpha \in D_F$. By the same argument as in i), we would conclude $\alpha \in D'$. So, $\alpha_0 \in D'$ and the theorem in the case of prime power indices would have been proved.

To prove that β_1 is actually in A' , consider the centralizer A^* of Ω^* in A . It contains of course the element β_1 . Now, it is easy to construct a field of degree l^{s-1} over Ω^* , which splits D_F .⁷ So, Ω^* is isomorphic to a subfield Ω_2 of D_F of degree l over F . Let D_2 be the centralizer of Ω_2 in D_F . Then that of Ω_2 in A is $A_2 = M_{(1)} \times D_2$. Since A_2 is conjugate to A^* in A , A_2 contains an element β_2 conjugate to β_1 . To prove $\beta_1 \in A'$, it suffices to prove $\beta_2 \in A'$.

Let $v_2 = N_{A_2/\Omega_2}(\beta_2)$. Let S^{**} be the set of all primes in F , which are not relatively prime to v_2 in Ω_2 . Let S'_∞ be the set of all real primes in F , where D_F is ramified. Put $S = S_0 + S^{**} + S'_\infty + \bar{S}$.

We are going to construct a cyclic extension K_2 of degree l^{s+1} over F , applying the theorem in [4] with $\tilde{C} = C = \Omega_2$. Since q remains prime in the l -component of $F(\xi_s)$, but splits completely in Ω_2 , Ω_2 is independent with $F(\xi_s)$ over F .

For $p \in S$, we define K_2^p and $(\alpha_p, K_2^p/F_p)$ as follows:

(a₂) for $p \in S_0 + \bar{S}$, $K_2^p = \Gamma^p$ and $(\alpha_p, K_2^p/F_p) = (\alpha_p, \Gamma^p/F_p)$.

(b₂) for $p \in S^{**}$, $K_2^p = F_p$,

(c₂) for $p \in S'_\infty$, K_2^p is complex and $(\alpha_p, K_2^p/F_p)$ is defined in the only way possible.

Since $\Omega_2^p = F_p$ for $p \in S^{**}$, $\Omega_2^p \subset K_2^p$ for $p \in S^{**}$. As before, we prove that, when $l = 2$ and when $F(\xi_s)$ is not cyclic over F ,

$$\prod_{p \in S} ((\sec 2\pi/2^{t+1})^{2^s}, K_2^p/F_p) = 1,$$

and that condition (d) is satisfied. Consequently, there exists a cyclic extension K_2 of degree l^{s+1} over F having the following properties:

(A₂) $\Omega_2 \subset K_2$,

(B₂) the p -adic completion of K_2 is K_2^p for $p \in S$,

(C₂) if K_2 is ramified at any prime p outside S , p remains prime in Ω_2 .

Again, we prove that there exists an element γ_2 in K_2 such that $N_{K_2/\Omega_2}(\gamma_2) = v_2$.

⁷ For example, let $\eta \in \Omega^*$ be totally negative and divisible exactly by the first power of \mathfrak{P} for $\mathfrak{P} \in S_0$. Then $\Omega^*(\eta^{1/a})$, where $a = l^{s-1}$, splits D_F .

Since K_2 splits D_F and therefore D_2 , it is a maximal subfield of A_2 and therefore of A . We have $N_{A_2\Omega_2}(\gamma_2) = N_{K_2\Omega_2}(\gamma_2) = v_2 = N_{A_2\Omega_2}(\beta_2)$, i. e., $N_{A_2\Omega_2}(\gamma_2) = N_{A_2\Omega_2}(\beta_2)$. By the hypothesis of induction β_2 differs from γ_2 only by an element in $A'_2 \subset A'$. Now,

$$N_{AF}(\gamma_2) = N_{\Omega_2 F}(v_2) = N_{AF}(\beta_2) = N_{AF}(\beta_1) = N_{\Omega_1 F}(v_1) = N_{D_F F}(\alpha) = 1,$$

i. e., $N_{AF}(\gamma_2) = 1$. But γ_2 is contained in the cyclic maximal subfield K_2 of A . So, γ_2 is a commutator in A and $\beta_2 \in A'$.

iii) $n = l_1^{s_1} l_2^{s_2} \cdots l_r^{s_r}$, general. Let F_i be an extension of degree m_i of k such that $(m_i, l_i) = 1$, and that D_{F_i} is of index $l_i^{s_i}$ over F_i . Let $\alpha_0 \in D$ be of norm 1. Then $\alpha_0 \in D'_{F_i}$. By Lemma 3, $\alpha_0^{m_i} \in D'$. But, by Lemma 4, $\alpha_0^n \in D'$. Since $(m_1, m_2, \cdots, m_r, n) = 1$, $\alpha_0 \in D'$. The theorem is thus completely proved.

If, in iii), the fields F_i are chosen such that $(m_i, l_i^{s_i}) = l_i^{s_i-1}$ and that D_{F_i} is of index l_i , we obtain $\alpha_0^{n/(l_1 l_2 \cdots l_r)} \in D'$. Since i) was proved for general algebras, we have the following

COROLLARY. *Let A be any simple algebra. Then, $N\alpha = 1$ in A implies $\alpha^{n/(l_1 \cdots l_r)} \in A'$, $n = l_1^{s_1} l_2^{s_2} \cdots l_r^{s_r}$ being the index of A . In particular, if the index of A is square free, any element of norm 1 in A is in A' .*

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ON THE REPRESENTATION OF SURFACES.*

By LAMBERTO CESARI.

1. Let S be any continuous surface of xyz -space and

$$(1) \quad S: x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \in Q \equiv (0, 1, 0, 1),$$

any representation of S upon the fundamental square Q . Let us call

$$\Phi_1: \quad y = y(u, v), \quad z = z(u, v),$$

$$(2) \quad \Phi_2: \quad z = z(u, v), \quad x = x(u, v), \quad (u, v) \in Q,$$

$$\Phi_3: \quad x = x(u, v), \quad y = y(u, v),$$

the relative plane transformations which we get by projecting S upon the three coordinate planes. T. Radó introduced the notion of a plane transformation essentially of bounded variation (eBV) and this concept is equivalent to the notion of a transformation of bounded variation (BVC) introduced by the author, as T. Radó has proved recently [T. R. 1, 3]. In addition the author proved [L. C. 2] that *the surface S has finite Lebesgue area if and only if the three plane transformations (2) are BVC (or eBV).*

Each plane transformation BVC (or eBV)

$$\Phi: \quad x = x(u, v), \quad y = y(u, v), \quad (u, v) \in Q,$$

admits, almost everywhere (a. e.) in Q , an (absolute) generalized Jacobian $J(u, v)$ according to the equivalent definitions that T. Radó and the author have introduced [T. R. 1, L. C. 1, 3]. Such a Jacobian is L -integrable on Q . If the functions $x(u, v)$, $y(u, v)$ have first partial derivatives or, at least, an asymptotic regular differential, a. e. in a set $M \subset Q$, then we have [T. R. 1, L. C. 3] a. e. in M

$$J(u, v) = |\partial(x, y)/\partial(u, v)|.$$

that is, the generalized Jacobian coincides, a. e. in M , with the ordinary Jacobian.

If the surface S has finite Lebesgue area, then the three transformations (2) are BVC and therefore have generalized Jacobians $J_1(u, v)$, $J_2(u, v)$, $J_3(u, v)$ a. e. in Q and these are L -integrable functions in Q . In addition we have [T. R. 1, L. C. 3]

* Received March 18, 1949.

$$(3) \quad L(S) \geq \iint_Q [J_1^2(u, v) + J_2^2(u, v) + J_3^2(u, v)]^{1/2} du dv,$$

that is the Lebesgue area is always greater than or equal to the classical integral calculated with generalized Jacobians.

T. Radó and the author [T. R. 1, L. C. 3] introduced equivalent notions of an absolutely continuous plane transformation (*eAC*, or *ACC*, according to the nomenclature of T. Radó) and showed, at the same time and independently, that in relation (3) the equality sign holds if and only if the three plane transformations (2) are *eAC* (or *ACC*).

Now the two following problems, that may be termed representation problems [see T. Radó 1, V, 2, 19, p. 475], arise:

I) Given a surface S of finite Lebesgue area, does there exist such a representation (1) for which equality holds in relation (3), that is, for which the Lebesgue area is given by the classical integral calculated with generalized Jacobians?

II) Given a surface S of finite Lebesgue area, does there exist a representation (1) for which the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ have ordinary first partial derivatives a. e. in Q and for which equality holds in relation (3), that is, the Lebesgue area is given by the classical integral calculated with ordinary Jacobians?

In recent papers [L. C. 5, 6, 7] the author gave an affirmative answer to the problem I. In the present paper an affirmative answer is given to the problem II. This problem has been called to the attention of the writer in the course of a correspondence with J. W. T. Youngs.

2. A partial affirmative answer to the problems I and II was first given by E. J. McShane [3] for saddle surfaces and then, more generally, by C. B. Morrey [1] for open nondegenerate surfaces. The theorem of C. B. Morrey, for which the author gave a direct proof by means of the direct method of Calculus of Variations [L. C. 4], asserts the following:

Each continuous, open nondegenerate surface S of finite Lebesgue area admits a representation (1) such that:

a) *the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ are of bounded variation and absolutely continuous in the sense of Tonelli in Q ;*

b) *the first partial derivatives x_u, x_v, \dots, z_v , that exist a. e. in Q as a consequence of a), are L^2 -integrable functions in Q ;*

c) *a. e. in Q we have $E = G$, $F = 0$, where $E = x_u^2 + y_u^2 + z_u^2$, $G = x_v^2 + y_v^2 + z_v^2$, $F = x_u x_v + y_u y_v + z_u z_v$;*

d) the Lebesgue area $L(S)$ is given by the classical integral:

$$L(S) = \iint_Q [(\partial(x, y)/\partial(u, v))^2 + \dots]^{1/2} du dv \\ = \iint_Q (EG - F^2)^{1/2} du dv = \frac{1}{2} \iint_Q (E + G) du dv.$$

3. A point P of Q is said to be an *exceptional point* [L. C. 5] for the representation (1) of the surface S if a neighborhood $U(P)$ of P exists such that the surface defined by (1) upon $U(P)$ has zero Lebesgue area. A set $I \subset Q$ is said to be an *exceptional set* [L. C. 5] if I is open (relative to Q) and all points of I are exceptional. A set $M \subset Q$ is said to be a *proper set* [L. C. 5] if its complementary set $M - Q$ is exceptional.

Let $f(x)$, $a \leq x \leq b$, be a function, continuous in (a, b) , and $E \subset (a, b)$ a closed set. We shall define the total variation $V(f, E, \alpha, \beta)$ of $f(x)$ on the set E in the interval (α, β) , $a \leq \alpha < \beta \leq b$, as the least upper bound of

$$\sum_{i=1}^{n-1} |f(x_i) - f(x_{i+1})|$$

for each set $\alpha \leq x_1 < x_2 < \dots < x_n \leq \beta$ of points of E . If in (α, β) there is no point of E or only one, we put $V(f, E, \alpha, \beta) = 0$. We shall say that $f(x)$ is of *bounded variation* V upon E in (a, b) if $V(f, E, a, b) < +\infty$ [Cf. S. Saks 1, p. 221]. We shall say that $f(x)$ is *absolutely continuous* V upon the set E in (a, b) if, for any $\epsilon > 0$ there is a $\delta > 0$ such that, for each set of intervals (α_i, β_i) , $i = 1, 2, \dots, n$, of (a, b) of which no two have common points and $\sum_{i=1}^n (\beta_i - \alpha_i) < \delta$, we have

$$\sum_{i=1}^n V(f, E, \alpha_i, \beta_i) < \epsilon.$$

Let $f(x, y)$ be a continuous function in the fundamental square

$$Q = [0 \leq x \leq 1, 0 \leq y \leq 1]$$

and $M \subset Q$ a closed point set. For each \bar{x} , $0 \leq \bar{x} \leq 1$, let us call $M(\bar{x})$ the linear closed set (possibly empty) of the points (\bar{x}, y) , $0 \leq y \leq 1$, of M . The linear set $M(\bar{y})$ has an analogous definition.

For each \bar{x} , $0 \leq \bar{x} \leq 1$, let $V_y(f, M, Q, \bar{x})$ denote the total variation of the function of y alone, $f(\bar{x}, y)$, $0 \leq y \leq 1$, on the closed set $M(\bar{x})$ in the interval $0 \leq y \leq 1$. $V_x(f, M, Q, \bar{y})$ has an analogous definition. We shall say that $f(x, y)$ is of *bounded variation* V on the set M in the square Q [L. C. 5], if $V(f, M, Q, x)$, $0 \leq x \leq 1$, which is a function of x alone, is finite a. e. and L -integrable, and if the corresponding fact holds for $V_x(f, M, Q, y)$, $0 \leq y \leq 1$. We shall say that $f(x, y)$ is *absolutely continuous*

V on the set M in the square Q [L. C. 5], if $f(x, y)$ is of bounded variation V on the set M in the square Q , if for almost all \bar{x} , $0 \leq \bar{x} \leq 1$, the function of y alone, $f(\bar{x}, y)$, $0 \leq y \leq 1$, is absolutely continuous V on $M(\bar{x})$ in $0 \leq y \leq 1$, and if the same fact holds when x and y are interchanged. If $f(x, y)$ is continuous in Q and of bounded variation V on a closed set M of Q , then $f(x, y)$ admits first partial asymptotic derivatives a. e. in M and also an asymptotic differential [L. C. 7, p. 306]. If we consider that M may be a totally disconnected closed set, it is evident that, in general, the ordinary first partial derivatives and even the regular asymptotic differential, may not exist.

4. Let S be any continuous surface of finite Lebesgue area. The author proved that there always exists a representation of S upon the fundamental square for which the area is given by the classical integral calculated with generalized Jacobians (§ 1, problem I). More precisely, the author proved the following:

THEOREM A. *Each continuous surface S of finite Lebesgue area admits a representation (1) such that:*

a) *the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ are of bounded variation V and absolutely continuous V on a closed set $M \subset Q$, where M is a proper set for the representation (1);*

b) *each of the functions $x(u, v)$, $y(u, v)$, $z(u, v)$, which, as a consequence of a) has first partial asymptotic derivatives and an asymptotic differential a. e. in M , also has a regular asymptotic differential and the first partial asymptotic derivatives are L^2 -integrable on M ;*

c) *a. e. in M we have $E = G$, $F = 0$;*

d) *the generalized Jacobians J_1, J_2, J_3 are zero at every point of $Q - M$ (the exceptional set) and coincide with the ordinary ones a. e. in M (the proper set).*

e) *the Lebesgue area $L(S)$ of S is given by the classical integral*

$$\begin{aligned} L(S) &= \iint_Q (J_1^2 + J_2^2 + J_3^2)^{\frac{1}{2}} du dv \\ &= \iint_M (EG - F^2)^{\frac{1}{2}} du dv = \frac{1}{2} \iint_M (E + G) du dv. \end{aligned}$$

This theorem was proved by the author in recently published papers [L. C. 6, 7] and given in the preliminary note [5] already mentioned.

5. First of all we notice that in the Theorem A, the statement b) can be replaced by the following stronger one:

b') the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ have ordinary first partial derivatives a. e. in M , and therefore also regular asymptotic differentials, and the first partial derivatives are L^2 -integrable in M .

This stronger statement was proved by the author in the paper [7] already mentioned, but is not given in the preliminary note [5].

In the present paper we shall deduce from Theorem A, with statement b) replaced by b'), the following Theorem B which gives an affirmative answer to problem II (§ 1).

THEOREM B. *Each continuous surface S of finite Lebesgue area admits a representation (1) upon the fundamental square Q such that*

a*) the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ have first partial derivatives a. e. in Q ;

b) the derivatives x_u, x_v, \dots, z_v are L^2 -integrable in Q ;

c) a. e. in Q we have $E = G, F = 0$;

d) the Lebesgue area $L(S)$ of S is given by the classical integral;

e) the functions x, y, z are of bounded variation V and absolutely continuous V in a closed set $M \subset Q$, where M is a proper set for the representation (1);

f) the first partial derivatives x_u, x_v, \dots, z_v are zero a. e. in the exceptional set $Q - M$.

Remarks. Theorem B holds for each continuous surface and is similar to the theorem of Morrey 2 for open nondegenerate surfaces, but the conclusion a*) of Theorem B is weaker than the conclusion a) of the theorem of Morrey. We observe that in the theorem of Morrey 2 d) is a consequence of a) and b) [see T. Radó, 1, V, 2, 26, p. 480]. In Theorem B d) is not a consequence of a*) and b). E. J. Mickle [1] has proved that there exists a continuous surface of finite Lebesgue area that has no representation (1) with x, y, z functions of bounded variation in the sense of Tonelli in Q . This important result of Mickle proves that in Theorem B conclusion a*) cannot be replaced by the stronger conclusion a).

Proof of Theorem B.

6. For open nondegenerate surfaces Theorem B, with $M \equiv Q$, coincides with the theorem of C. B. Morrey. Let S be a surface of the type A [L. C. 5, 6; base surface according to C. B. Morrey (1)]. Then Theorem A holds with the following further conclusions [L. C. 6]:

f) M consists of the boundary Q^* of Q and of a finite or countable set of disjoint circles $\bar{C}_i \subset Q$, $i = 1, 2, 3, \dots$, and any limit points of these circles lie on the boundary Q^* of Q ;

g) on the circles C_i equations (1) define closed nondegenerate surfaces S_i .

By a), b'), c) equations (1) give upon each circle C_i a representation of the surface S_i which satisfies the theorem of C. B. Morrey and also the Theorem B. Therefore we have simply to modify the representation (1) only on the exceptional set $Q - M$. Let us divide Q into 4 equal squares each of side-length 2^{-1} and of these let us choose only those that are, with their boundaries, completely interior to $Q - M$; let us call them q_1, q_2, \dots, q_r . Let us divide each of the remaining squares into 4 equal squares, each of the side-length 2^{-2} and of these let us choose only those that are completely interior to $Q - M$; let us call them q_{r+1}, \dots, q_{r_2} , and so on. We get an infinite sequence of squares $q_1, q_2, \dots, q_j, \dots$, which fill $Q - M$ and

$$Q - M = \sum_{j=1}^{\infty} q_j, \quad |Q - M| = \sum_{j=1}^{\infty} |q_j|.$$

We know [see the Appendix] that on each of these squares q_j , there exists a particular plane transformation of q_j into itself

$$\phi_j: u = u_j(\alpha, \beta), \quad v = v_j(\alpha, \beta), \quad (\alpha, \beta) \in q_j,$$

such that ϕ_j is continuous, monotone, identical on the boundary q_j^* of q_j and constant on a countable set of disjoint squares $q_{jk} \subset q_j$, $k = 1, 2, 3, \dots$, which are completely interior to q_j and for which we have

$$\sum_{k=1}^{\infty} |q_{jk}| = |q_j|, \quad j = 1, 2, \dots.$$

Let us now put

$$X(\alpha, \beta) = \begin{cases} x[u_j(\alpha, \beta), v_j(\alpha, \beta)] & \text{if } (\alpha, \beta) \in q_j, \quad j = 1, 2, \dots, \\ x(\alpha, \beta) & \text{if } (\alpha, \beta) \in M, \end{cases}$$

and analogously let us define $Y(\alpha, \beta)$ and $Z(\alpha, \beta)$. The equations

$$(4) \quad S: x = X(\alpha, \beta), \quad y = Y(\alpha, \beta), \quad z = Z(\alpha, \beta), \quad (\alpha, \beta) \in Q,$$

give a new representation of the surface S , which coincides with (1) on M .

In a.e. interior point of the circles C_i , $i = 1, 2, \dots$, and hence a.e. in M , we have $X_u = x_u, \dots, Z_v = z_v$. Let (α, β) be an interior point of one of the squares q_{jk} , $k = 1, 2, \dots, j = 1, 2, \dots$. Then the functions u and v are constant on q_{jk} and therefore the functions X, Y, Z are also constant there which implies $X_u = 0, \dots, Z_v = 0$. This holds a.e. in $Q - M$. Indeed

$$|Q - M| = \sum_{j=1}^{\infty} |q_j| = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |q_{jk}|.$$

This completes the verification of statements a*) and f). In addition we have $E = G, F = 0$ a. e. in M and $E = G = F = 0$ a. e. in $Q - M$. Therefore c) holds. Statements d) and e) follow from Theorem A.

Theorem B for surfaces of type A is now completely proved.

Note. The procedure that we have used here is related to a simpler observation of J. W. T. Youngs [3; see also T. R. 1, V, 2, 21, p. 476].

7. Let S be a closed nondegenerate surface. Then Theorem A holds with the following further conclusions [L. C. 7, p. 342]:

f') M consists of a closed circle \bar{C} completely interior to Q , or of the boundary C^* of circle C and a finite or countable set of disjoint circles $\bar{C}_i \subset C$, $i = 1, 2, 3, \dots$, where any limit points of these circles lie on C^* ;

g') on the circle C , or on the circles C_i , equations (1) define closed nondegenerate surfaces S_i .

We have now only to repeat the procedure of 6.

8. Let S be any continuous surface. Then Theorem A holds but M is now any closed set and furthermore it may be totally disconnected. Let us point out that, while the first partial asymptotic derivatives on M of the functions x, y, z can be calculated by using only values of these functions on M , the ordinary first partial derivatives on M must be calculated by using values of these functions on $Q - M$ as well. In this general case also we modify on $Q - M$ the given representation (1) of Theorem A, but we must be careful not to lose the existence of the first partial derivatives at points of M where they exist (that is, a. e. in M).

Let $\{P, P'\}$, $\{I, I'\}$ denote the distance between two points, or two sets; let $\delta(I)$ be the diameter of the set I . For each $0 < \delta \leq (2)^{\frac{1}{2}}$ set

$$\omega(\delta) = \max \{S(P), S(P')\}$$

for each pair P, P' of points of Q such that $\{P, P'\} \leq \delta$. Of course $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$.

As noted in Section 6 there are squares $p_1, p_2, \dots, p_n, \dots$ non overlapping and such that $Q - M = p_1 + p_2 + \dots$. For $n = 1, 2, \dots$, divide p_n into a finite number of congruent squares $p_{n,1}, p_{n,2}, \dots, p_{n,i_n}$ such that

$$\omega(\text{diam } p_{n,i}) < \{p_n, M\}^2 \leq \{p_{n,i}, M\}^2, \quad i = 1, 2, \dots, i_n.$$

Call these squares $q_1, q_2, \dots, q_n, \dots$. Then for $n = 1, 2, \dots$,

$\alpha)$ q_n and its boundary q_n^* lie in $Q - M$;

$\beta)$ $\omega(\text{diam } q_n) < \{q_n, M\}^2$;

$\gamma)$ $Q - M = q_1 + q_2 + \dots + q_n + \dots$.

Repeating the procedure of 6, we get a new representation

$$(4) \quad S: x = X(\alpha, \beta), y = Y(\alpha, \beta), z = Z(\alpha, \beta), (\alpha, \beta) \in Q,$$

of the surface S . Let us prove that the functions X, Y, Z have first partial derivatives a. e. in M , and precisely at the points of M where the functions x, y, z have them.

Let P' be a point of $Q - M$ and consequently a point of a square q_i . Let \bar{P} be any point of q_i^* . We have

$$|X(P') - X(\bar{P})|, |x(P') - x(\bar{P})| \leq \omega(\text{diam } q_i), \quad X(\bar{P}) = x(\bar{P})$$

and the same relations for Y and Z . Hence

$$(5) \quad |X(P') - x(P')| \leq 2\omega(\text{diam } q_i), \dots$$

Now let $P \equiv (\alpha, \beta)$ be a point of M at which the first partial derivative x_u exists; let $P' \equiv (\alpha + h, \beta)$, $h \neq 0$, be any point of Q on the straight line $v = \beta$.

If $P' \equiv (\alpha + h, \beta)$ is a point of M , then

$$[X(\alpha + h, \beta) - X(\alpha, \beta)]/h = [x(\alpha + h, \beta) - x(\alpha, \beta)]/h.$$

If $P' \equiv (\alpha + h, \beta)$ is not contained in M , then P' lies in a square q_i and we have, from (5) and β),

$$\begin{aligned} [X(\alpha + h, \beta) - X(\alpha, \beta)]/h &= [x(\alpha + h, \beta) - x(\alpha, \beta)]/h \\ &\quad + [X(P') - x(P')]/h, \\ |[X(P') - x(P')]/h| &\leq 2[\omega(\text{diam } q_i)/\{q_i, M\}] < 2\{q_i, M\} \\ &< 2\{P', M\} \leq 2\{P', P\} = 2|h|. \end{aligned}$$

In any case

$$\begin{aligned} \lim_{h \rightarrow 0} [X(\alpha + h, \beta) - X(\alpha, \beta)]/h \\ = \lim_{h \rightarrow 0} [x(\alpha + h, \beta) - x(\alpha, \beta)]/h = x_u(\alpha, \beta), \end{aligned}$$

that is, the first partial derivative X_u exists and $X_u(\alpha, \beta) = x_u(\alpha, \beta)$.

The same holds for all other first partial derivatives of the functions

X, Y, Z . This is sufficient to prove that the representation (4) of the surface S satisfies all the conditions of Theorem B.

Consequence of Theorem B.

9. The notion of integral upon a surface

$$J_S = \iint_Q F(x, y, z, H_1, H_2, H_3) du dv$$

as a Lebesgue integral has been studied by E. J. McShane [1, 2, 3] for oriented surfaces S that are given by a representation (1) satisfying conditions a) and b) of 2.

More generally T. Radó [2] studied the same notion under the hypotheses that the functions x, y, z have first partial derivatives a. e. in Q and that the Lebesgue area $L(S)$ is finite and given by the classical integral calculated with ordinary Jacobians.

The notion of the integral J_S for any continuous oriented surface S of finite Lebesgue area given by any representation (1) was introduced recently by the author as a Weierstrass integral. The author [8] proved that

1) J_S is independent of the representation of the oriented surface S ;

2) if $\|S_n, S\| \rightarrow 0$ in the sense of Fréchet and if $L(S_n) \rightarrow L(S)$, then $J_{S_n} \rightarrow J_S$.

3) J_S is equal to the Lebesgue integral (6) with generalized or ordinary (relative) Jacobians H_1, H_2, H_3 for each representation (1) for which the area is given by the classical integral. From Theorem A we know that there is always at least one such representation.

From Theorem B that we have just proved, we know that each surface S of finite Lebesgue area has at least one representation for which the functions x, y, z have ordinary partial derivatives a. e. and for which the Lebesgue area is given by the classical integral. Therefore, by choosing such a representation, each integral J_S can always be calculated as a Lebesgue integral (6) with ordinary Jacobians, that is, as one of the integrals studied by T. Radó [2].

Appendix.

There exist continuous plane transformations Φ of a square q into itself, monotone on q , identical upon the boundary q^ of q , constant upon a countable set of non overlapping closed squares $q_i \subset q$, $i = 1, 2, \dots$, $\sum |q_i| = |q|$.*

α) *Proof.* Let us indicate by E_3 the subdivision of a square into 9

equal squares by means of straight lines parallel to the sides. Let us perform E_3 upon q and let q_0 be the closed central square. Let us perform E_3 upon each of the remaining 8 squares and let $q_{i_1 0}$ be the 8 central squares, and so on. By repeating this procedure indefinitely we have a countable set of closed non-overlapping squares $q_0, q_{i_1 0}, q_{i_1 i_2 0}, \dots, q_{i_1 i_2 \dots i_{n-1} 0}, q_{i_1 i_2 \dots i_n 0}, \dots, i_1, i_2, \dots, i_n, \dots = 1, 2, \dots, 8$, contained in q (Sierpinski, 1). Let K be the collection of all these squares plus all individual points of q not on such squares. K is an upper semicontinuous decomposition of q into continua separating neither q , nor the plane and therefore (R. L. Moore, 1) there exists a continuous transformation f of K into the closed unit circle C . f is a biunique transformation between C^* and q^* . There exists a transformation ϕ of C into q , biunique, which coincides with f^{-1} upon C^* . The continuous monotone transformation $\Phi = \phi f$ of q into itself is identical on q^* , is constant upon the squares $q_{i_1 i_2 \dots i_n 0}$ and $\sum_n \sum_{i_1 i_2 \dots i_n} |q_{i_1 i_2 \dots i_n 0}| = |q|$.

β) *An elementary construction of a particular transformation Φ .*

Let $p = A_1 A_2 A_3 A_4$ be any convex plane quadrilateral. Let d be the diameter of p , O be the center of gravity of the four vertices (of the mass unity), let $p_0 = A'_1 A'_2 A'_3 A'_4$ be the image of p under the similarity transformation with center O and ratio $1/n$ ($n \geq 3$ integer). Let us divide the sides of p into three equal parts by means of the points $B_i, i = 1, 2, \dots, 8$, ordered upon p^* starting from A_1 and in the same order as the points A_i . Let us indicate by E_n the subdivision of p into the 9 quadrilaterals $p_0, p_1 = A_1 B_1 A'_1 B_8, p_2 = B_1 B_2 A'_2 A'_1, p_3 = B_2 A_2 B_3 A'_2, p_4 = B_3 B_4 A'_3 A'_2, p_5 = B_4 A_3 B_5 A'_3, p_6 = B_5 B_6 A'_4 A'_3, p_7 = B_6 A_4 B_7 A'_4, p_8 = B_7 B_8 A'_1 A'_4$. It is elementary to prove that the 9 quadrilaterals p_i are all convex and have diameters $d_i \leq (3/4)d, i = 0, 1, \dots, 8$. If p is a square, then the subdivision E_3 considered in α) is a particular case of E_n .

Let q, q' be the squares $q \equiv [0 \leq \alpha \leq 1, 0 \leq \beta \leq 1], q' \equiv [0 \leq u \leq 1, 0 \leq v \leq 1]$. For each integer $n \geq 3$ we define the transformation Φ_n as follows. We perform the subdivision E_3 upon the square q , n times, as in α , letting the central squares always remain undivided, and analogously we apply the subdivision E_n to the square q' , n times, letting the central quadrilaterals always remain undivided. We obtain in q certain squares $q_0, q_{i_1 0}, q_{i_1 i_2 0}, \dots, q_{i_1 i_2 \dots i_{n-1} 0}, q_{i_1 i_2 \dots i_n 0}$ of side length $1/3, 1/3^2, \dots, 1/3^n, 1/3^n$ and correspondingly in q' certain convex quadrilaterals $q'_0, q'_{i_1 0}, q'_{i_1 i_2}, \dots, q'_{i_1 i_2 \dots i_{n-1} 0}, q'_{i_1 i_2 \dots i_n}$ of diameters $\leq 2^{1/2}/n, 1/n(3/4)^{2^{1/2}}, 1/n(3/4)^{2^{1/2}}, \dots, 1/n(3/4)^{n-1} 2^{1/2}, (3/4)^n 2^{1/2}$ ($i_1, i_2, \dots, i_n = 1, 2, \dots, 8$). If δ_n is the maximum of such diameters then $\delta_n \rightarrow 0$ when $n \rightarrow \infty$. Let us divide each of the squares

and quadrilaterals of q and q' into four triangles by means of the two diagonals and let Φ_n be the continuous biunique transformation of q into q' which is linear in each triangle of q and which makes each triangle of q' correspond to the corresponding triangle of q . Let us observe that we have successively divided each side of q and the corresponding side of q' into three equal parts and each of these parts again into three equal parts and so on, and that Φ_n is linear in each part. Therefore Φ_n is identical on the boundary q^* of q . It is easy to prove that the sequence Φ_n , $n = 1, 2, \dots$, converges uniformly in q toward a transformation Φ having the required properties.

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ON NON-LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER.*

By PHILIP HARTMAN and AUREL WINTNER.

It was shown in [7] and [1] that, under suitable conditions on the coefficient function $f=f(x)$ of the differential equation $y''+f(x)y=0$, one can conclude the existence of at least one solution $y=y(x)$ representable as a Laplace-Stieltjes transform of a non-decreasing function,

$$(1) \quad y(x) = \int_0^{\infty} e^{-tx} d\phi(t), \quad (d\phi(t) \geq 0).$$

The first part of the present paper deals with the question of existence of solutions representable in the form (1) in the case of differential equations which, in contrast to the type mentioned before, are of first order but non-linear. In the particular case of a Ricatti differential equation, such a result was obtained in [8]. The general results to be proved below admit of various applications, including one to an implicit equation $f(x, y) = 0$. The unrestricted nature of the solution of the differential equation, that is, the existence of a solution $y=y(x)$ in the large (i. e., on some half-line, say $0 < x < \infty$) is, of course, part of the problem considered.

The second part of the paper deals with conditions which assure the non-existence of unrestricted solutions (and, what is more, with the existence of a constant $L > 0$ having the property that no solution can be defined on an interval whose length exceeds L). If $f(x, y)$ is a continuous function on the (x, y) -plane and $y=y(x)$ is a solution of the differential equation $y'=f(x, y)$ on an interval $a < x < b$, which cannot be continued onto any interval $a < x < b + \epsilon$, where $\epsilon > 0$, then $|y(x)| \rightarrow \infty$ as $x \rightarrow b - 0$; cf. [4]. The relation between a solution $y=y(x)$ and a corresponding number b will be considered for differential equations $y'=f(x, y)$, which possess no unrestricted solutions.

Part I.

1. A function $y=y(x)$ defined on an interval is called completely monotone there if it possesses derivatives of arbitrarily high order which satisfy

$$(2) \quad (-1)^n y^{(n)}(x) \geq 0 \text{ for } n = 0, 1, \dots$$

* Received May 20, 1949.

and for all values of x on this interval. According to the Hausdorff-Bernstein theorem, a function $y(x)$ possesses a representation (1) for $0 < x < \infty$ in terms of a monotone, but not necessarily bounded, function $\phi(t)$ on the half-line $t \geq 0$ if and only if $y(x)$ is completely monotone on the half-line $0 < x < \infty$. Thus the discussion of existence of solutions (1) of a differential equation is reduced to a discussion of the existence of completely monotone solutions.

(I) Let $F(x, y)$, $G(x, y) > 0$ be defined and of class C^∞ for $x > 0$, $y > 0$ and let their partial derivatives satisfy

$$(3) \quad (-1)^{j+k} F / \partial x^j \partial y^k \geq 0 \text{ for } j, k = 0, 1, 2, \dots$$

and

$$(4) \quad (-1)^{j+k} G / \partial x^j \partial y^k \geq 0 \text{ for } (j, k) \neq (0, 0).$$

Then every solution $y = y(x)$ (> 0) of

$$(5) \quad F + Gy' = 0$$

is completely monotone on any interval on which it exists.

It may be remarked that (5) can be written in the form $F/G + y' = 0$ and (I) can then be applied to the case where F, G are replaced by $F/G, 1$ respectively. The wording of (I) and this application of (I) lead to different criteria for (5) to possess only completely monotone solutions (that is, it is possible for the functions F, G to satisfy conditions (3), (4) whether or not the functions $F/G, 1$ satisfy them). This will be shown in 3.

(I) leads to the following result concerning implicit equations:

COROLLARY 1_I. Let $f(x, y)$ be defined and of class C^∞ for $x > 0$ and $y > 0$ and let its partial derivatives satisfy

$$(6) \quad (-1)^{j+k} f / \partial x^j \partial y^k \geq 0 \text{ for } (j, k) \neq (0, 0), (0, 1), \text{ while } \partial f / \partial y > 0.$$

In addition, let

$$(7) \quad -\infty \leq \lim_{y \rightarrow 0} f(x, y) < 0 \text{ and } 0 < \lim_{y \rightarrow \infty} f(x, y) \leq \infty$$

for every fixed $x > 0$. Then there exists a unique solution $y = y(x)$ of the implicit equation

$$(8) \quad f(x, y) = 0$$

for $0 < x < \infty$, and this $y = y(x)$ is completely monotone for $0 < x < \infty$.

If the conditions of (I) are modified slightly, then certain solutions of (5) are representable in the form (1), where $\phi(t)$ is monotone and bounded on $0 \leq t < \infty$.

COROLLARY 2₁. Let $F(x, y), G(x, y) > 0$ be defined and of class C^∞ for $x \geq 0, y \geq 0$ and such as to satisfy (3), (4), respectively. In addition, let

$$(9) \quad F(x, 0) \equiv 0 \text{ for } 0 \leq x < \infty.$$

Then the solution of (5) determined by an initial condition

$$(10) \quad y(0) = y_0 (\geq 0)$$

exists on the entire half-line $0 \leq x < \infty$ and is completely monotone there (in particular, if $y_0 > 0$, then $y(x) > 0$ for all $x \geq 0$).

It may be remarked that if $y = y(x)$ is any solution of (5), under the conditions of Corollary 2₁, then there exists a number $a \geq 0$ such that the definition of $y = y(x)$ can be extended over a half-line $a < x < \infty$ and $y(x)$ is completely monotone there. Consequently, $y(x)$ has a representation of the form (1) valid for $a < x < \infty$, where $\phi(t)$ is monotone (not necessarily bounded) on the half-line $0 < t < \infty$.

Another type of modification of (I) leads to cases in which a solution of (5) is not completely monotone but is the primitive function of a completely monotone function (that is, $y'(x)$ is completely monotone).

(II) Let $F(x, y), G(x, y) > 0$ be defined and of class C^∞ for $x > 0, y > 0$ and let their partial derivatives satisfy

$$(11) \quad (-1)^{j+k} \partial^{j+k} F / \partial x^j \partial y^k \geq 0 \text{ for } j, k = 0, 1, 2, \dots,$$

and

$$(12) \quad (-1)^{j+k+1} \partial^{j+k} G / \partial x^j \partial y^k \geq 0 \text{ for } (j, k) \neq (0, 0).$$

Then every solution $y = y(x) (> 0)$ of

$$(13) \quad F - Gy' = 0$$

possesses a completely monotone derivative $y'(x)$ on the interval on which $y(x)$ is defined.

In contrast to the situation in (I), the assertion of (II) is in its most inclusive form when $G \equiv 1$; that is, if F, G satisfy the conditions of (I) and F, G are replaced by $F/G, 1$ respectively, then (II) is still applicable (but the converse is obviously false).

Corresponding to the Corollary 1_I, there results the following corollary of (II):

COROLLARY 1_{II}. If $f(y)$, where $0 < y < \infty$, is an increasing function of class C^∞ satisfying either

$$(14) \quad (-1)^k d^k f / dy^k \geq 0 \text{ for } k = 2, 3, \dots$$

or, more generally,

$$(15) \quad (-1)^k d^k (df/dy)^{-1} / dy^k \geq 0 \text{ for } k = 0, 1, 2, \dots$$

(which means that $(df/dy)^{-1}$ is completely monotone), then the inverse function $y = y(x)$ of $x = f(y)$ possesses a completely monotone first derivative $y'(x)$.

Also, if the conditions of (II) are modified so as to assure the existence of unrestricted solutions of (13), an analogue of Corollary 2_I is obtained.

COROLLARY 2_{II}. Let $F(x, y)$, $G(x, y) > 0$ be defined and of class C^∞ for $x \geq 0$, $y > 0$ and let their partial derivatives satisfy (11), (12), respectively. In addition, let there exist positive continuous functions $\lambda = \lambda(s)$ and $\phi = \phi(s)$ defined for $0 < s < \infty$ satisfying

$$(16) \quad F(x, y) / G(x, y) \leq \lambda(x) \phi(y)$$

and

$$(17) \quad \int_0^\infty \lambda(s) ds < \infty \text{ and } \int_0^\infty ds / \phi(s) = \infty.$$

Then the solution $y = y(x)$ of (13) determined by the initial condition

$$(18) \quad y(0) = y_0 (> 0)$$

exists on the entire half line $0 \leq x < \infty$ and possesses a completely monotone first derivative $y'(x)$.

Remarks similar to those following the Corollary 2_I are applicable to the last corollary also.

2. Proof of (I). Let $y = y(x)$ be a solution of (5) on an x -interval S ; so that $y(x) > 0$ on S (since F and G are defined only for $x > 0$, $y > 0$). Let (2_n) denote the inequality in (2) for a fixed n . Thus $y(x) > 0$ implies (2_0) . Also, (5) and $F \geq 0$, $G \geq 0$ imply (2_1) . Suppose that (2_0) , (2_1) , \dots , (2_n) hold on S . It must be verified that (2_{n+1}) holds on S .

The chain rule for differentiation shows that, if $n > 0$, the n -th derivative of $F(x, y(x))$ consists of a sum of terms of the type

$$(19) \quad (\partial^{j+k} F / \partial x^j \partial y^k) (dy/dx)^\alpha (d^2 y / dx^2)^\beta \cdots (d^n y / dx^n)^\nu$$

where

$$(20) \quad j + \alpha + 2\beta + \cdots + n\nu = n.$$

This is clear for the case $n = 1$, since $dF(x, y(x))/dx = \partial F / \partial x + (\partial F / \partial y) \times (dy/dx)$. Assume that the statement is correct for a given $n \geq 1$. Then a differentiation of (19) shows that the statement is correct for $n + 1$. For the derivative of the first factor in (19) is

$$\partial^{j+k+1} F / \partial x^{j+1} \partial y^k + (\partial^{j+k} F / \partial x^j \partial y^{k+1}) (dy/dx);$$

so that the sums (20) belonging to the corresponding two terms are $(j + 1) + \alpha + 2\beta + \cdots = n + 1$ and $j + (\alpha + 1) + 2\beta + \cdots = n + 1$, respectively. The derivative of a factor, say $(d^e y / dx^e)^\epsilon$, is $\epsilon(d^e y / dx^e)^{\epsilon-1} (d^{e+1} y / dx^{e+1})$, so that the sum (20) corresponding to this term is $j + \alpha + 2\beta + \cdots + [e(\epsilon - 1) + (e + 1)] + \cdots$ or $j + \alpha + 2\beta + \cdots + (e\epsilon + 1) + \cdots = n + 1$. Hence, by (3) and the induction hypothesis on $(2_0), \dots, (2_n)$, the inequality $(-1)^n d^n F(x, y(x)) / dx^n \geq 0$ holds on S .

Similarly, the chain and product rules show that the n -th derivative of $G(x, y(x))y'(x)$ is a sum of $Gy^{(n+1)}$ and terms of the type

$$(\partial^{j+k} G / \partial x^j \partial y^k) (dy/dx)^\alpha (d^2 y / dx^2)^\beta \cdots (d^n y / dx^n)^\nu$$

where $j + \alpha + 2\beta + \cdots + n\nu = n + 1$. Hence, by (3) and the induction hypothesis,

$$(-1)^n (d^n (G(x, y(x))y'(x)) / dx^n - G(x, y(x))d^{n+1}y / dx^{n+1}) \geq 0.$$

If (5) is differentiated n times, the result can be written in the form $Gd^{n+1}y / dx^{n+1} = -Q$, where $(-1)^n Q \geq 0$. Since $G > 0$, it follows that (2_{n+1}) holds. This completes the proof of (I).

Proof of Corollary 1_I. The condition $\partial f / \partial y > 0$ and (7) imply the existence of a unique solution $y = y(x) > 0$ of (8) for $0 < x < \infty$. Clearly, $y = y(x)$ is of class C^∞ and satisfies the differential equation (5), where $F(x, y) = \partial f / \partial x$ and $G(x, y) = \partial f / \partial y$. Since $G = \partial y / \partial y > 0$, it only remains to show that conditions (3), (4) are satisfied. Since

$$(-1)^j \partial^{j+k} F / \partial x^j \partial y^k = (-1)^j \partial^{j+1+k} f / \partial x^{j+1} \partial y^k \geq 0$$

and

$$(-1)^{j+1} \partial^{j+k} G / \partial x^j \partial y^k = (-1)^{j+1} \partial^{j+1+k} f / \partial x^j \partial y^{k+1} \geq 0$$

by (6), Corollary 1_I follows from (I).

Proof of Corollary 2_I. In virtue of (I), the Corollary 2_I will be proved if it is shown that a solution of (5) defined on some interval $0 \leq a \leq x \leq b$ can be extended over the half line $a \leq x < \infty$. The smoothness (C^∞) of F and G implies that if such an extension is possible, then it is necessarily unique by virtue of "local uniqueness." That is, if $x_0 \geq 0$, $y_0 \geq 0$, then there exist some number $\delta > 0$ and a unique solution of (5) on $x_0 \leq x \leq x_0 + \delta$ satisfying $y(x_0) = y_0$. This is also true for $y_0 = 0$; in fact, by (9), such a solution is given by $y(x) \equiv 0$, while the smoothness of F and G assures the uniqueness of this solution. In particular, if a solution $y = y(x)$ of (5) vanishes for some value of $x = x_0 \geq 0$, then $y \equiv 0$.

Consider an arbitrary solution $y = y(x)$ (≥ 0) of (5) on some interval $0 \leq a \leq x \leq b$. If $y(x)$ vanishes at some point of this interval, it vanishes identically and possesses the continuation $y(x) \equiv 0$ for $a \leq x < \infty$. If $y(x) > 0$ on the interval, then $F \geq 0$ and $G > 0$ imply that $y'(x) \leq 0$, by (5). Thus, $y(x)$ possesses either an extension over the half-line $a \leq x < \infty$ or over some interval $a \leq x < x_0$, where $(x, y(x))$ tends to a boundary point of the domain of definition of F and G as $x \rightarrow x_0$ (that is, $y(x) \rightarrow 0$ holds as $x \rightarrow x_0$). In the latter case, if $y(x_0)$ is defined to be zero, the function $y = y(x)$ becomes a solution of (5) on the closed interval $a \leq x \leq x_0$, and $y(x_0) = 0$. But then $y(x) \equiv 0$, which contradicts $y(x) > 0$ on $a \leq x \leq b$. Consequently, $y(x)$ possesses a continuation over the half-line $a \leq x < \infty$ (and $y(x)$ does not vanish on this half-line). This completes the proof of Corollary 2_I.

Proof of (II). The proof of this statement is similar to that of Theorem (I) and can be omitted.

Proof of Corollary 1_{II}. If (14) holds, then $df/dy > 0$ for $0 < y < \infty$, by virtue of the strict monotony of f . For if $df/dy = 0$ when $y = y_0 > 0$, then $df/dy = 0$ when $0 < y \leq y_0$, since $df/dy \geq 0$ and $d^2f/d^2y \geq 0$, by (14). But this contradicts the assumption that $f(y)$ is strictly monotone. The assumption (15) is meaningless unless $df/dy > 0$ for $0 < y < \infty$. Consequently in either case, the inverse function $y = y(x)$ of $x = f(y)$ possesses a continuous first derivative and satisfies the differential equation $1 - (df/dy)y' = 0$. Corollary 1_{II} now follows from (II) by considering F, G in (13) to be $(df/dy)^{-1}, 1$, respectively. In the general case, where (15) is assumed, (11) reduces to (15) if $j = 0$, and to $0 \geq \dot{0}$ if $j > 0$. Thus, (II) is applicable and the proof of Corollary 1_{II} is complete.

Proof of Corollary 2_{II}. In virtue of (II), it is sufficient to show that if $y = y(x)$ is a solution of (13) defined on some interval $a \leq x \leq b$, then the

definition of $y(x)$ can be extended over the half-line $a \leq x < \infty$. But this follows from a known result [5], p. 451. [Strictly speaking, [5] becomes applicable only if $F(x, y)/G(x, y)$ is defined for $0 \leq x < \infty$ and $-\infty < y < \infty$. But if it is noticed that (13) implies that $y'(x) \geq 0$, so that $y(x) \geq y(a) > 0$ for all $x > a$ for which $y(x)$ is defined, it is clear that the domain of definition of F/G , namely, $x \geq 0, y > 0$, is sufficient for the purposes at hand.]

3. It was mentioned above that if (5) is written in the form

$$(5') \quad F/G + y' = 0$$

and (I) is applied to (5'), where F, G are replaced by $F/G, 1$, respectively, then the resulting criteria for (5) and/or (5') to have only completely monotone solutions neither imply nor are implied by (I) itself. That is to say, (3) and (4) neither imply nor are implied by

$$(3') \quad (-1)^j \partial^{j+k}(F/G)/\partial x^j \partial y^k \geq 0 \text{ for } j, k = 0, 1, \dots$$

In order to see this, first let $F(x, y) = e^y$ and $G(x, y) = y$; so that $G > 0$ and (3), (4) are satisfied when $y > 0$ but (3') is not (e. g., if $j = 0$ and $k = 1$, then $\partial(F/G)/\partial y = d(y^{-1}e^y)/dy = e^y y^{-1}(1 - y^{-1}) < 0$ if $y < 1$). Next, let $F(x, y) \equiv 1$ and $G(x, y) = e^{-y}$. Then $G > 0$, and (3') is satisfied (since $F/G = e^y$) but (4) is not (e. g., if $j = 0$ and $k = 2$, then $(-1)^2 \partial^2 G/\partial y^2 = -e^{-y} < 0$).

It may be observed that, although (14) implies (15), the converse is not true. For example, $f(y) = e^y$ satisfies (15) but not (14).

Part II.

4. This part will be concerned with the existence or non-existence of unrestricted solutions of differential equations, that is, of solutions existing on some half-line $a \leq x < \infty$. The assumption (9) in Corollary 2_I and the assumption involving (16), (17) in Corollary 2_{II} merely play the rôle of assuring that all solutions of (5) and (13), respectively, are unrestricted. These conditions can be replaced by other types of conditions which assure the existence of only some unrestricted solutions. In this direction, the following will be proved:

(III) Let $f(x)$ be a continuous function defined for large positive x such that

$$(21) \quad f(x) \geq 0$$

and that

$$(22) \quad r'' + f(x)r = 0$$

is non-oscillatory (e. g., let

$$(23) \quad \limsup x^2 f(x) < 1/4$$

as $x \rightarrow \infty$). Let $F(x, y)$ be defined and continuous for $x > 0$, $y > 0$ and satisfy

$$(24) \quad 0 \leq F(x, y) < y^2 + f(x)$$

on some half-strip $X \leq x < \infty$, $0 < y \leq \epsilon$, where $X > 0$, $\epsilon > 0$. Then the differential equation

$$(25) \quad F(x, y) + y' = 0$$

possesses some unrestricted solutions; in particular, if X is sufficiently large and if $y = y(x)$ is a solution of (25) and satisfies the initial condition

$$(26) \quad y(x_0) = y_0, \text{ where } x_0 \geq X \text{ and } y_0 \geq \epsilon,$$

then all continuations of $y = y(x)$ can be extended over the half-line $x_0 \leq x < \infty$.

This theorem can be combined with (I) to assure the existence of solutions of (5) which are representable in the form (1) on some half-line $x_0 \leq x < \infty$.

The differential equation (22) is said to be non-oscillatory if one (hence every) solution $r = r(x) \not\equiv 0$ of (22) has only a finite number of zeros on the half-line $X \leq x < \infty$ (for sufficiently large X). That (23) is sufficient for (22) to be non-oscillatory is a result of A. Kneser [2], p. 415. The last part of (III) involving "continuations" of a solution is needed, since the conditions on $F(x, y)$ do not imply the local uniqueness of solutions of (25).

The theorem (III) has an analogue involving the assumption that (22) is oscillatory and the conclusion that no solution of (25) is unrestricted.

(IV) Let $f(x)$, a continuous function defined for $0 \leq x < \infty$, be such as to make (22) oscillatory (e. g., let

$$(27) \quad \liminf x^2 f(x) > 1/4$$

as $x \rightarrow \infty$). Let $F(x, y)$ be defined and continuous for $x > 0$, $-\infty < y < \infty$ and let it satisfy

$$(28) \quad F(x, y) > y^2 + f(x) \quad (X \leq x < \infty, -\infty < y < \infty).$$

Then no solution of (25) is unrestricted.

The fact that (27) implies that (22) is oscillatory (i. e., not non-oscilla-

tory) was proved by Kneser [2], p. 415. One can modify (IV) so as to assure the existence of a constant $L > 0$ with the property that no solution of (25) is defined over an interval $(0 <) x_0 < x < x_0 + L$ of length L .

COROLLARY 1_{IV}. Suppose that $f(x)$, $F(x, y)$ satisfy the conditions of (IV) and that the distance between successive zeros of any non-trivial solution $r = r(x)$ of (22) does not exceed a fixed $L > 0$ (e. g., let

$$(29) \quad f(x) \geq \pi^2/L^2$$

for $0 \leq x < \infty$). If $y = y(x)$ is a solution of (25) satisfying an initial condition $y(x_0) = y_0$, then to every continuation of $y(x)$ there corresponds a number b with the property that

$$(30) \quad y(x) \rightarrow -\infty \text{ as } x \rightarrow b - 0$$

and $x_0 < b < x_0 + L$.

It may be remarked that if (25) is replaced by

$$(31) \quad y' = F(x, y)$$

then (IV) remains valid; the same is true of its Corollary 1_{IV} except that (30) must be replaced by

$$(32) \quad y(x) \rightarrow \infty \text{ as } x \rightarrow b - 0.$$

Corollary 1_{IV} can be generalized as follows:

COROLLARY 2_{IV}. Let $f(x)$, $g(y)$ be defined and continuous for $0 \leq x < \infty$, $-\infty < y < \infty$, respectively, and suppose that

$$(33) \quad f(x) \geq c > 0, \quad g(y) \geq 0$$

and

$$(34) \quad L = \int_{-\infty}^{\infty} dy/(g(y) + c) < \infty.$$

Let $F(x, y)$ be defined and continuous for $0 \leq x < \infty$, $-\infty < y < \infty$ and such as to satisfy

$$(35) \quad F(x, y) \geq f(x) + g(y).$$

Then the conclusions of Corollary 1_{IV} are valid.

Conditions for the non-existence of unrestricted solutions of (25) can take a very different form. An application of the theorem in [6], p. 554 for the differential equation $dx/dy = 1/F(x, y)$ implies the following result:

(V) Let $F(x, y)$ be defined and continuous for $0 \leq x < \infty$, $0 \leq y < \infty$ and let there exist two positive, continuous functions $\lambda(s)$, $\phi(s)$ on the half-line $0 < s < \infty$ satisfying

$$(36) \quad F(x, y) \geq \lambda(x)\phi(y)$$

and

$$(37) \quad \int_0^\infty ds/\lambda(s) < \infty \text{ and } \int_0^\infty \phi(s) ds = \infty.$$

Then no solution of (31) is unrestricted; furthermore, there belongs to every number $b > 0$ at least one solution $y = y(x) = y(x; b)$ of (31) on $0 \leq x < b$ satisfying (32).

The solutions $y = y(x; b)$ of this theorem can be made unique by adapting the conditions of the theorems of [3] to the situation at hand; cf. [3], p. 131.

5. *Proof of (III).* Let $r = r(x) \not\equiv 0$ be a solution (22). Then, since (22) is supposed to be non-oscillatory, $r(x)$ does not vanish for sufficiently large x , say for $x > X$. It can be supposed that $r(x) > 0$ for large x (for otherwise r can be replaced by $-r$). Then (21) implies that the graph of $r = r(x)$ is convex downwards for large x . Hence $r > 0$, $r' \geq 0$, $r'' \leq 0$, since r has no zeros for large x . Let $z = r'/r$. Then z satisfies the Ricatti differential equation

$$(38) \quad z' + z^2 + f(x) = 0 \quad (z = r'/r)$$

for $x > X$. Clearly, $z > 0$ and $z' \leq 0$. Hence, $z(x)$ tends to a limit $z(\infty) \geq 0$ as $x \rightarrow \infty$. In fact, $z(\infty) = 0$; for if $z(\infty) > 0$, then (21) and (38) show that $z' < -\text{const.}$, where $\text{const.} > 0$, for large x , which contradicts $z \geq 0$ for $x > X$.

For the given $\epsilon > 0$, let X be so large that $0 < z(x) < \epsilon$ if $x \geq X$. Consider a solution $y = y(x)$ of (25) and the initial condition (26), so that $y(x_0) \geq \epsilon > z(x_0)$.

The first inequality in (24) and the differential equation (25) show that $y'(x) \leq 0$. Hence every continuation $y(x)$ for increasing x can be extended over the half-line $x_0 \leq x < \infty$, unless there exists a number $b > x_0$ such that $y(x)$ is defined for $x_0 \leq x < b$ and $y(x) \rightarrow 0$ as $x \rightarrow b - 0$. It will be shown that such a number b cannot exist; in fact, $y(x) > z(x) > 0$, where $x \geq x_0$, holds for every continuation of $y(x)$. For suppose, if possible, that such is not the case for some continuation. Then

$$\epsilon \geq y(x) > z(x) \text{ for } x_1 \leq x < x_2 \text{ but } y(x_2) = z(x_2)$$

holds for a pair of numbers $x_1 (\geq x_0)$ and x_2 . According to (25) and (38),

$$(39) \quad y'(x) - z'(x) = z^2(x) + f(x) - F(x, y(x)).$$

Hence,

$$y'(x_2) - z'(x_2) = y^2(x_2) + f(x_2) - F(x_2, y(x_2)) > 0,$$

by (24). Consequently $0 = y(x_2) - z(x_2) > y(x_2 - \delta) - z(x_2 - \delta) > 0$ for sufficiently small $\delta > 0$. This contradiction shows that every continuation of $y(x)$ is unrestricted, which completes the proof of (III).

Proof of (IV). The assumption that (22) is oscillatory implies that if $z = z(x)$ is a solution of (38) on some interval $(X \leq) x_0 \leq x \leq x_1$, then, for a suitable choice of x_1 , it follows that $z(x) \rightarrow -\infty$ as $x \rightarrow x_1 - 0$. In order to see this, let $r(x)$ be the solution of (22) determined by the initial conditions $r(x_0) = 1$ and $r'(x_0) = z(x_0)$. Then the local uniqueness of solutions of (38) implies that $z(x) = r'(x)/r(x)$ holds on any interval on which $z(x)$ exists. If x_1 is chosen to be the first zero of $r(x)$ to the right of x_0 , then $z(x) \rightarrow -\infty$ as $x \rightarrow x_1 - 0$.

Suppose, if possible, that (25) has an unrestricted solution $y = y(x)$ on $(X \leq) x_0 < x < \infty$. Consider the solution $z = z(x)$ of (38) determined by the initial condition $z(x_0) = y(x_0)$. It will be shown that

$$y(x) < z(x) \text{ for } x_0 < x < x_1,$$

which will imply that $y(x) \rightarrow -\infty$ as $x \rightarrow b - 0$ for some $b \leq x_1$. This will contradict the assumption that $y(x)$ is unrestricted.

Since (25) and (38) imply (39), it follows from the assumption (28) that $y' - z' < 0$ at $x = x_0$; and so $y(x) < z(x)$ if $x > x_0$ is sufficiently near x_0 . Suppose that the relation in the last formula line fails to hold. Let $x_2 (> x_0)$ be the first value of x (to the right of x_0) at which it does not hold, so that $y(x_2) = z(x_2)$. Then it is seen from (39) and (28) that $y' - z' < 0$ at $x = x_2$. Since $y(x_2) - z(x_2) = 0$, it follows that $y(x) - z(x) > 0$ if $x (< x_2)$ is sufficiently near x_2 . This is obviously a contradiction and completes the proof of (IV).

Proof of Corollary 1_{IV}. The proof of this assertion is a consequence of the proof just completed, since the assumption on the zeros of a solution $r = r(x)$ of (22) implies that the numbers x_0, x_1 (of the last proof) satisfy $x_1 - x_0 < L$.

The parenthetical part of Corollary 1_{IV}, that which asserts that (29) is sufficient to assure that the distance between successive zeros of a solution

$r = r(x)$ of (22) does not exceed L , follows from Sturm's comparison theorem if the harmonic oscillator, $r'' + (\pi^2/L^2)r = 0$, is used as a "minorant" for (22).

Proof of Corollary 2IV. Let $y = y(x)$ be a solution of (25) on some interval $(0 \leq) x_0 \leq x < b$. Then (33), (35) and the differential equation (25) imply that $y'(x) < 0$ and that $1 = -y'/F(x, y(x)) \leq -y'/(f(x) + g(y(x)))$, hence $1 \leq -y'/(g(y(x)) + c)$. Let the latter inequality be integrated from $x = x_0$ to $x = b$. If the monotone function $y = y(x)$ is introduced as a new variable in the integral on the right, one obtains

$$b - x_0 \leq \int_{\alpha}^{\beta} dy/(g(y) + c), \text{ where } \alpha = y(b - 0), \beta = y(x_0).$$

It follows therefore from (34) that $b - x_0 < L$.

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ON THE SPECTRA OF TOEPLITZ'S MATRICES.*

By PHILIP HARTMAN and AUREL WINTNER.

1. If f_n , where $n = 0, \pm 1, \dots$, is a sequence of complex numbers satisfying

$$(1) \quad f_n = \bar{f}_{-n} \text{ and } \sum_{n=-\infty}^{\infty} |f_n|^2 < \infty,$$

let L , T , H denote the Laurent, Toeplitz, Hankel matrices defined by $L = (f_{n-m})$, where $n, m = 0, \pm 1, \dots$, and $T = (f_{n-m})$ and $H = (f_{n+m+1})$, where $n, m = 0, 1, \dots$, respectively. Thus an L -matrix consists of two (identical) T -matrices on its main diagonal and two (complex-conjugate) H -matrices on its counter-diagonal.

Let x^+ denote a vector (x_0, x_1, \dots) and x^- a vector (\dots, x_{-2}, x_{-1}) , finally (x^-, x^+) a vector $(\dots, x_{-1}, x_0, x_1, \dots)$. When no misunderstanding is possible, x will be used to represent a vector of any of the types x^+ , x^- , (x^-, x^+) . The symbol $|x|$ will denote the length of x . The second condition in (1) implies that $y^+ = Tx^+$, $y^- = Hx^-$ and $(y^-, y^+) = L(x^-, x^+)$ are defined (without being necessarily of finite length) whenever the x -vector is of finite length.

The problem of determining the spectrum of L was solved by Toeplitz ([3], [4]; cf. [2], pp. 152-155) and is comparatively simple. According to the Fischer-Riesz theorem, there exists a (unique, real-valued) function $f(\theta)$ of class (L^2) on $(0, 2\pi)$, having $\sum f_n e^{in\theta}$ as its Fourier series,

$$(2) \quad f(\theta) \sim \sum_{n=-\infty}^{\infty} f_n e^{in\theta}.$$

In terms of this $f(\theta)$, the situation can be described as follows: The matrix L is bounded in Hilbert's sense if and only if the function $f(\theta)$ is essentially bounded, that is, $|f(\theta)| \leq \text{const.}$ for almost all θ on $(0, 2\pi)$. Furthermore, the number λ belongs to the spectrum, S_L , of L if and only if the measure of the set of θ -values on $(0, 2\pi)$ satisfying $|f(\theta) - \lambda| < \epsilon$ is of positive measure for every fixed $\epsilon > 0$. Similarly, the point spectrum, P_L , contains a given λ if and only if the measure of the set of θ -values satisfying $f(\theta) - \lambda = 0$ is of positive measure. In other words, S_L is identical with the spectrum of the distribution function of $f(\theta)$, and P_L with the set of

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discontinuity points of this distribution function. In fact, the resolvent of L is the Laurent matrix belonging to the function $(\lambda - f(\theta))^{-1}$. This is readily verified, since Parseval's relation implies that $y = Lx$ is equivalent to $f(\theta)x(\theta) = y(\theta)$, where $x(\theta) \sim \sum x_n e^{in\theta}$ and $y(\theta) \sim \sum y_n e^{in\theta}$.

All of this depends on the formal circumstance that the product of two L -matrices is again an L -matrix (and that this multiplication is simply isomorphic to the multiplication of the respective functions f). Correspondingly, because no such rule of multiplication holds for T -matrices, we could find nothing in the literature on the location of their spectra.

The purpose of the following considerations is to fill somewhat this gap. The results are anything but of a final nature. They reach far enough to show that the spectral situation for T -matrices is quite different from that for L -matrices. The case of the H -matrices, which is again different, will be considered at the end of the paper.

2. According to Toeplitz,

(I) T is bounded if and only if L is; or, equivalently, if and only if the function (2) is bounded (almost everywhere).

This is easily verified as follows: Let $L_j = (f_{n-m})$, where $n, m = j, j+1, \dots$, be a section (in the lower right-hand corner) of L . Then L is bounded if and only if L_j is bounded for every fixed j and the sequence of numbers representing the norms of the matrices L_j is bounded for $j = 0, \pm 1, \dots$. Since L_j is identical with T (except that its domain is a "different" Hilbert space), the assertion of (I) follows.

If S_L, S_T denote the spectra of T, L , respectively, (I) might suggest that S_T is identical with S_L . In fact, the proof of (I) implies that the least and greatest points of S_T coincide with those of S_L . Further circumstantial evidence results by observing that, on the one hand, the finite sections ${}_jT = (f_{n-m})$, where $n, m = 0, 1, \dots, j$, and ${}_jL = (f_{n-m})$, where $n, m = -j, -j+1, \dots, j$, of T and L , respectively, satisfy ${}_{2j+1}T = {}_jL$ and that, on the other hand, a necessary condition for λ to be in S_T [in S_L] is the existence of a sequence of numbers $\lambda_1, \lambda_2, \dots$, having the property that λ_j is an eigenvalue of ${}_jT$ [of ${}_jL$], and $\lambda_j \rightarrow \lambda$ as $j \rightarrow \infty$. The difficulty is that this necessary condition is not sufficient, as is shown by an example of Toeplitz representing the matrix of the quadratic form $\sum 2x_{2n}x_{2n+1}$. Actually this example is neither a T - nor an L -matrix; however, the results below (cf. (II)) imply that there exist an L -matrix and a sequence of numbers $\lambda_1, \lambda_2, \dots$ satisfying the above conditions, while the limit point λ is not in S_L .

It turns out that the above guess is wrong. In this direction, it will be proved that

(II) S_T contains S_L but need not be identical with S_L .

In view of the preceding remarks, the first part of (II) implies the following criterion:

(III) If the function (2) is continuous on $(0, 2\pi)$, then S_T is identical with S_L (which is an interval unless $f = \text{const.}$)

3. A little more probing into the question seems to indicate that S_T is always an interval (not only for a continuous f) unless $f = \text{const.}$; more specifically, that S_T is connected with the spectrum of the distribution function of that (bounded) harmonic function $u(r, \theta)$ on the unit circle $r < 1$ which, in terms of (1), satisfies $u(r, \theta) \rightarrow f(\theta)$ as $r \rightarrow 1 - 0$ for almost all θ . Correspondingly, it seems that T , in contrast to L , cannot possess a point spectrum unless $f = \text{const.}$ (almost everywhere).

We were unable to prove either of these conjectures. All that will be proved in their favor is contained in the following facts:

(IV) If T is not a constant multiple of the unit matrix, and if λ is in the point spectrum of L , then λ is not in the point spectrum of T . Moreover, the least and greatest points of S_T (if they are distinct, i. e., if $f \neq \text{const.}$) are not in the point spectrum of T .

It can be expected that the above conjectures are provable in the case of "smooth" functions (1). In this direction, it will be shown that

(*) T cannot have a point spectrum if

$$(3) \quad F(z) = \sum_{n=-\infty}^{+\infty} f_n z^n \neq \text{const.} \quad (f_n = \bar{f}_{-n})$$

is a rational function of z . (Needless to say, since (2) is of class (L^2) , the rational function (3) has no pole on the unit circle $|z| = 1$.)

4. The proofs proceed as follows.

Proof of (II). Let λ be a point of S_L . It is no loss of generality to suppose that $\lambda = 0$. For, if $I = (\delta_{nm})$, where $n, m = 0, \pm 1, \pm 2, \dots$ or $n, m = 0, 1, \dots$, is the unit matrix, then $L - \lambda I$ is also an L -matrix, and $T - \lambda I$ is the corresponding T -matrix. Since $\lambda = 0$ is in S_L , there belongs to every $\epsilon > 0$ a point $x = x_\epsilon = (\dots, x_{-1}, x_0, x_1, \dots)$ of Hilbert's space satisfying $|x| = 1$ and $|Lx| < \epsilon$. But $|x^j| \rightarrow |x|$ and $|L(0, x^j)| \rightarrow |Lx|$ as

$j \rightarrow \infty$, if $x^j = (x_{-j}, x_{-j+1}, \dots)$. Hence, $|x^j| > 1 - \epsilon$ and $|Tx^j| < 2\epsilon$ for sufficiently large j . Consequently, $\lambda = 0$ is in the spectrum of T . This proves the first part of (II).

The second part of (II) follows from the first part of (IV). For example, if (1) is a step-function which assumes only a finite set of distinct values $\lambda_1, \lambda_2, \dots, \lambda_n$, then S_L is a pure point spectrum consisting of the eigenvalues $\lambda = \lambda_j$, where $j = 1, \dots, n$ (and each eigenvalue has an infinite multiplicity). By (II), the point λ_j is in S_T . But $\lambda = \lambda_j$ is not in the point spectrum of T , by (IV). Hence, $\lambda = \lambda_j$ is not an isolated point of S_T . Since $\lambda = \lambda_j$ is an isolated point of S_L (if multiplicities are ignored), the second part of (II) follows.

Proof of (IV). If x^*, y^* are defined as at the beginning of Section 1, it is clear that $(y^-, y^+) = L(0, x^*) = (Hx^*, Tx^*)$, and that Parseval's relation implies the identity $f(\theta)x^*(\theta) = y^*(\theta) + y^-(\theta)$ (for almost all θ). Thus, if λ is in the point spectrum of T and x^* is a corresponding eigenvector, then

$$y^*(\theta) = \lambda x^*(\theta) \text{ and } (f(\theta) - \lambda)x^*(\theta) = y^-(\theta), \text{ where } y^-(\theta) \sim \sum_{n=1}^{\infty} y_{-n} e^{-in\theta}.$$

Let λ be in the point spectrum of L ; it can be supposed that $\lambda = 0$. Then $f(\theta)$ vanishes on a set of positive measure. Suppose, if possible, that λ is in the point spectrum of T and that $x^* \neq 0$ is an eigenvector satisfying

$$(4) \quad f(\theta)x^*(\theta) = y^-(\theta).$$

Accordingly, $y^-(\theta)$ vanishes on a set of positive measure on $(0, 2\pi)$. It follows therefore from a theorem of F. and M. Riesz, conjectured by Fatou, that $y^-(\theta) \equiv 0$, since $y^-(-\theta)$ is of class (L) and is the boundary function of the power series $\sum y_{-n} z^n$ on the unit circle $|z| < 1$. Thus $f(\theta)x^*(\theta) \equiv 0$, and so $x^*(\theta) = 0$ when $f(\theta) \neq 0$. Since $x^* \neq 0$, it follows that $x(\theta)$ cannot vanish on a set of positive measure on $(0, 2\pi)$. Hence $f(\theta) \equiv 0$. But then T is the zero matrix, which contradicts the assumption that T is not a constant multiple of the unit matrix. Consequently, $\lambda = 0$ is not in the point spectrum of T . This proves the first part of (IV).

In order to prove the second part of (IV), it can be supposed that $\lambda = 0$ is the least point of S_T , and therefore of S_L . Thus $f(\theta) \geq 0$ for almost all θ . Suppose that $\lambda = 0$ is in the point spectrum of T and let $x^* \neq 0$ denote an eigenvector satisfying (4). The Fourier expansions of $x^*(\theta)$, $y^-(\theta)$ show that they are orthogonal,

$$\int_0^{2\pi} \overline{x^*(\theta)} y^-(\theta) d\theta = 0; \text{ consequently, } \int_0^{2\pi} f(\theta) |x^*(\theta)|^2 d\theta = 0,$$

by (4). Since $f(\theta) \geq 0$ and $x^*(\theta) \equiv 0$, this leads to the same contradiction as before.

Proof of ().* Since $f_n = \bar{f}_{-n}$, the function (3) satisfies the functional equation $F(1/z) = F(z)$. Let N be the order of the pole, if any, of $F(z)$ at $z = 0$. It will first be shown that $F(z)$ has at least $2N$ zeros on the z -plane.

Only the case $N > 0$ need be considered. The rational function $F(z)$ can be expressed in the form $F(z) = P(z)/z^N Q(z)$, where $P(z)$, $Q(z)$ are (relatively prime) polynomials of degree j , k , respectively, satisfying $P(0) \neq 0$, $Q(0) \neq 0$. It is easy to see that $j \geq 2N$. In fact, since

$$F(1/z) = z^N P(1/z)/Q(1/z) = z^{N-j+k} \{z^j P(1/z)\} / \{z^k Q(1/z)\},$$

and since $z^j P(1/z)$, $z^k Q(1/z)$ are polynomials which do not vanish at $z = 0$, it follows from $F(z) = F(1/z)$ that $N - j + k = -N$, hence $j = 2N + k \geq 2N$.

Suppose, if possible, that $\lambda = 0$ is in the point spectrum of T , and let $x^* \neq 0$ be an eigenvector satisfying (4). Then, formally,

$$(5) \quad F(z)X(z) = \sum_{n=1}^{\infty} y_n z^{-n}, \text{ where } X(z) = \sum_{n=0}^{\infty} x_n z^n.$$

Since $X(z)$ is regular for $|z| < 1$, the function $F(z)X(z)$ is meromorphic for $|z| < 1$. On the other hand, (5) shows that $F(z)X(z)$ has a regular analytic continuation for $0 < |z| < \infty$.

If $F(z)$ has no pole at $z = 0$, then (5) implies that $y_{-1} = y_{-2} = \dots = 0$. Since $F(z) \not\equiv 0$, it follows that $X(z) \equiv 0$. This contradicts $x^* \neq 0$.

Consequently, it can be supposed that $F(z)$ has at $z = 0$ a pole of order $N \geq 1$. Then

$$z^N F(z)X(z) = y_{-1} z^{N-1} + y_{-2} z^{N-2} + \dots + y_{-N}.$$

But since $z^N F(z)$ has at least $2N$ zeros, at least N zeros of $F(z)$ are in the circle $|z| \leq 1$.

Since the polynomial on the right-hand side in the last formula line has at most $N - 1$ zeros, $X(z)$ has at least one pole in the circle $|z| \leq 1$. Since $X(z)$ is regular for $|z| < 1$, the pole must be on $|z| = 1$. But this is impossible, since $x^*(\theta) = X(e^{i\theta})$ is of class (L^2) on $(0, 2\pi)$. This contradiction completes the proof of (*).

5. It is easy to show that (II) and (IV) contain the following corollary:

(α) *T cannot be completely continuous unless it is the zero matrix.*

In order to prove this assertion, (α) , suppose that (α) is false. Then, for some T distinct from the zero matrix, the set of the cluster points of S_T consists of the single point $\lambda = 0$. It follows therefore from (II) that S_L consists of a sequence $\lambda_1, \lambda_2, \dots$ satisfying $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$. Furthermore, not every λ_j is 0, since otherwise L , and therefore the corresponding T , is the zero matrix. Accordingly, some $\lambda_j \neq 0$ is in S_T but is not in the point spectrum of T , by (IV). Hence, T cannot be completely continuous.

It may be mentioned that the assertion of (α) holds for L -matrices also:

(β) L cannot be completely continuous unless it is the zero matrix.

In fact, (β) is a corollary of the following assertion:

(β^*) L cannot have a point spectrum containing an eigenvalue of finite multiplicity.

In order to prove the latter assertion, (β^*) , suppose that λ is a value contained in the point spectrum of L . This will be the case if and only if $\text{meas } E_\lambda > 0$, where E_λ denotes that subset of the interval $0 \leq \theta < 2\pi$ on which $f(\theta) = \lambda$. Correspondingly, it is readily seen from Parseval's relation that $x = (\dots, x_{-1}, x_0, x_1, \dots)$ is an eigenvector of L , belonging to the eigenvalue λ , whenever the corresponding function $x(\theta)$ vanishes on the complement of E_λ . Since the linear space of such functions $x(\theta)$ cannot be finite-dimensional, (β^*) follows.

For the Hankel matrices H , defined after (1), the situation is quite different, since (α) and (β) become contrasted by the following fact:

(γ) H must be completely continuous whenever the w -image of the circle $|z| < 1$ under the mapping

$$(6) \quad w(z) = \sum_{n=1}^{\infty} f_n z^n$$

is of finite area, i. e., $\sum_{n=1}^{\infty} n |f_n|^2 < \infty$.

In fact, a matrix (a_{nm}) must be completely continuous if $\sum \sum |a_{nm}|^2$ is convergent (Hilbert). Since the convergence of this double series is equivalent to the convergence of the simple series $\sum n |f_n|^2$ if $a_{nm} = f_{n+m-1}$, the assertion of (γ) follows from the definition of H .

6. In order to make the Hankel matrix $H = (f_{n+m-1})$ Hermitian, it will from now on be assumed that $\{f_n\}$ is a real sequence; cf. (1). Thus

$$(7) \quad g(\theta) \sim \sum_{n=1}^{\infty} f_n \cos n\theta \quad \text{and} \quad h(\theta) \sim \sum_{n=1}^{\infty} f_n \sin n\theta$$

are the real and imaginary parts of (6) on $|z| = 1$, where $z = e^{i\theta}$.

The following criterion goes back to Toeplitz ([3]; cf. [2], p. 154 and [1], p. 223):

(a) *H must be bounded whenever either of the functions (γ) is bounded (almost everywhere) for $0 \leq \theta \leq \pi$.*

This criterion, (a), suggests that (γ) in Section 5 admits of the following variant:

(b) *H must be completely continuous whenever either of the functions (γ) is continuous for $0 \leq \theta \leq \pi$.*

Clearly, the two sufficient criteria supplied by (b) are independent of each other, and of the criterion supplied by (γ) .

In order to prove (b), suppose, for instance, that the first of the functions (γ) is continuous for $0 \leq \theta \leq \pi$. Then there exist cosine polynomials $g_1(\theta), g_2(\theta), \dots$ which tend to $g(\theta)$ uniformly for $0 \leq \theta \leq \pi$. On the other hand, the norm of the Hankel matrix H belonging to $g(\theta)$ does not exceed $\text{const.} \max |g(\theta)|$, where the const. is an absolute constant (cf. [1], p. 223), and the correspondence between H and $g(\theta)$ is distributive. Consequently, if H_k denotes the Hankel matrix belonging to the trigonometric polynomial $g_k(\theta)$, then, since the uniformity of $g_k(\theta) \rightarrow g(\theta)$ means that $\max |g(\theta) - g_k(\theta)| \rightarrow 0$ as $k \rightarrow \infty$, it follows that the norm of $H - H_k$ tends to 0 as $k \rightarrow \infty$. Hence, H must be completely continuous if every H_k is. Finally, the complete continuity of every H_k follows from (γ) in Section 5, since the power series (6) belong to H_k , being a rational polynomial, is regular for $|z| \leq 1$.

7. We do not know what is necessary and sufficient for the boundedness of an Hermitian Hankel matrix. One part of the criterion, supplied by (I) in Section 2 for T -matrices, is true for H -matrices, the other is not. In fact, the situation is as follows:

(i) *H is bounded if, but not only if, the corresponding T (or L) is.*

First, since f_n is now supposed to be real, (1), (2) and (γ) show that $f(\theta) = f_0 + 2g(\theta)$, where f_0 is a constant. It follows therefore from (a) in Section 6 that if $f(\theta)$ is a bounded function, then H is a bounded matrix. Hence the positive part of the last italicized statement, (i), follows from (I) in Section 2. On the other hand, the negative part of (i) follows from (a) by choosing $f_n = n^{-1}$ in (γ) , since the function $h(\theta) = \sum n^{-1} \sin n\theta$ is, but the conjugate function $g(\theta) = \sum n^{-1} \cos n\theta$ is not, bounded for $0 < \theta < \pi$.

Because of the contrast between (I) and (i), the determination of S_H is even more difficult than that of S_T (cf. (II), (III) and (*) in Sections 2 and 3). In this direction, and in view of the guesses formulated before (IV), the proof of the following fact is not without interest:

(ii) $\lambda = 0$ is in the cluster spectrum of every (bounded, Hermitian) Hankel matrix.

It is understood that by the cluster spectrum of a bounded, Hermitian matrix is meant the λ -set consisting of the continuous spectrum and of the cluster values of the points of the point spectrum, with the proviso that an eigenvalue of infinite multiplicity is considered as a cluster value of the point spectrum. According to Weyl ([5], p. 378), the point $\lambda = 0$ is in the cluster spectrum of a bounded, Hermitian matrix H if and only if there exists in Hilbert's space a sequence of unit vectors x^0, x^1, x^2, \dots which tend weakly to the zero vector and are transformed by H into vectors Hx^0, Hx^1, Hx^2, \dots which tend strongly to the zero vector.

It follows that $\lambda = 0$ is sure to be in the cluster spectrum of H if $|He^k| \rightarrow 0$ as $k \rightarrow \infty$, where e^0, e^1, \dots denotes the sequence of vectors which form the successive columns of the unit matrix. Hence, in order to prove (ii), it is sufficient to ascertain that $|He^k| \rightarrow 0$ is true when H is a Hankel matrix, $H = (f_{n+m-1})$. But it is clear from the definition of e^k that, for such an H ,

$$|He^k|^2 = \sum_{n=k}^{\infty} f_n^2,$$

and so $|He^k| \rightarrow 0$ follows from (1).

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ON THE SOLUTIONS OF THE EQUATION OF HEAT CONDUCTION.*

By PHILIP HARTMAN and AUREL WINTNER.

There are obvious analogies between solutions of Fourier's equation

$$(1) \quad u_{xx} = u_t$$

on the open unit square

$$(2) \quad S: \quad 0 < x < 1, \quad 0 < t < 1$$

and solutions of Laplace's equation

$$(3) \quad u_{xx} + u_{tt} = 0$$

on the open unit circle

$$(4) \quad C: \quad x^2 + t^2 < 1.$$

These analogies deal with questions of existence, uniqueness, and limits at the boundary points of (2), (4), respectively. The object of this paper is to develop these analogies in a certain direction.

By a solution of (1) on (2) (or (3) on (4)) is meant a function for which u_{xx} , u_t (or u_{xx} , u_{tt}) exist and satisfy (1) (or (3)). In particular, u_x , u_t exist; however, this does not imply the continuity of u . What is true in this regard is that if u is a continuous solution of (3) on some open domain, then u is regular analytic on that domain. The corresponding statement for solutions of (1) is as follows:

(I) *If $u(x, t)$ is a continuous solution of (1) on an open domain, then $u(x, t)$ is of class C^∞ on that domain.*

This follows from (II) and (IV) below.

Let K denote the set of points on the three line segments which form the lower and lateral boundaries of S , so that

$$K = I' + J + I'',$$

where I' , J , I'' denote the respective segments

$$(5) \quad I': (x = 0, 0 \leq t < 1), \quad I'': (x = 1, 0 \leq t < 1).$$

$$J: (0 < x < 1, t = 0),$$

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Gevrey ([2], pp. 372-374) has proved the following theorem of uniqueness:

(II) *Let $u(x, t)$ be defined on $S + K$ and have the following properties: u is uniformly continuous on S , vanishes identically on K , and is a solution of (1) on S . Then u vanishes identically.*

Actually, Gevrey's wording of his theorem assumes the continuity of u_x and u_t , but a glance at his proof shows that this pair of additional assumptions is not used at all. Gevrey's proof of (II) depends on a refined, but still very elementary, form of the classical maximum principle, namely, on the following fact:

(III) *Let $u = u(x, t)$ be defined on the closure of S and have the properties that it is uniformly continuous and satisfies (1) on S . Then u assumes its maximum (for the closure of S) on K .*

The analogues of radial boundary limits for solutions of (3) on (4) are the 1-dimensional boundary limits,

$$(6_1) \quad \lim_{x \rightarrow +0} u(x, s) = f_1(s), \quad (6_2) \quad \lim_{x \rightarrow 1-0} u(x, s) = f_2(s),$$

and

$$(6_3) \quad \lim_{t \rightarrow +0} u(s, t) = f_3(s),$$

for the case of solutions of (1) on (2). These limits (when they exist) form the "boundary function" of $u(x, t)$ on the set K .

Let $\theta(x, t)$ denote the Jacobi ϑ_3 -function, with a suitable change of units on the axes of the independent variables,

$$(7) \quad \theta(x, t) = 1 + 2 \sum_{k=1}^{\infty} \exp(-k^2 \pi^2 t) \cos k\pi x.$$

Put

$$(8_1) \quad u_1(x, t) = - \int_0^t \theta_x(x, t-s) f_1(s) ds,$$

$$(8_2) \quad u_2(x, t) = \int_0^t \theta_x(x-1, t-s) f_2(s) ds,$$

and

$$(8_3) \quad u_3(x, t) = \frac{1}{2} \int_0^1 \{ \theta(x-s, t) - \theta(x+s, t) \} f_3(s) ds,$$

where $\theta_x = \partial\theta/\partial x$. The existence theorem mentioned above is as follows:

(IV) Let $f_1(s)$, $f_2(s)$, $f_3(s)$ form a uniformly continuous boundary function on K ; i. e., let each of these functions be continuous for $0 \leq s \leq 1$ and satisfy $f_1(0) = f_3(0)$ and $f_2(0) = f_3(1)$. Then (8_1) , (8_2) , (8_3) exist for every (x, t) on S (as Lebesgue integrals) and their sum,

$$(9) \quad u(x, t) = u_1(x, t) + u_2(x, t) + u_3(x, t),$$

represents a solution of (1) on S , is uniformly continuous on S and satisfies (6_1) -(6_3) for $0 \leq s \leq 1$.

This theorem seems to be between the lines of the classical literature but is not explicitly stated and proved. That (9) is a continuous solution of (1) on S , and satisfies (6_1) -(6_3) for $0 < s < 1$, is contained in standard facts; cf., e. g., [7], pp. 111-118. But this does not imply the important point in (IV), namely, the uniform continuity of (9); in this regard, cf. the remarks below concerning the function (10). Actually, the existence of a solution of (1) on S which is uniformly continuous on S and satisfies (6_1) -(6_3) for $0 \leq s < 1$ follows from the existence theorems in [5], pp. 214-229; [2], pp. 317-342. Furthermore, the representation (9) of this (unique) uniformly continuous solution can be verified by showing that the Green functions, whose existence is proved in [5], [2], reduce to the kernels in (8_1) -(8_3), when the domain in question is S .

A more direct verification of (IV) can be sketched as follows: Simple calculations show the truth of (IV) in the cases in which

$$f_1(s) = 1, \quad f_2(s) = 1, \quad f_3(s) = 1$$

for $0 \leq s \leq 1$;

$$f_1(s) = 0, \quad f_2(s) = 1, \quad f_3(s) = s;$$

for an arbitrary positive integer k ,

$$f_1(s) = 0, \quad f_2(s) = 0, \quad f_3(s) = \sin k\pi s;$$

finally,

$$f_1(s) = s^k, \quad f_2(s) = 0, \quad f_3(s) = 0.$$

These sets (f_1, f_2, f_3) , along with those which result if f_1 and f_2 (and correspondingly, s and $1-s$ in f_1 and f_2) are interchanged, form a Weierstrassian basis of all continuous functions on $K = I' + I'' + J$. The existence of a solution u of (1) on S , uniformly continuous on S and reducing to a given continuous boundary function on K , follows from the maximum principle, (III). The existence of this solution u and its representation (9) can be deduced from an adaptation of the maximum principle for functions having representations (9).

In fact, (9) can be written as

$$u(P) = \int_K G(P, Q) f(Q) dK,$$

where P and Q denote points of S and K , respectively, and $f(Q)$ denotes (f_1, f_2, f_3) , a continuous function on K . But $G(P, Q) \geq 0$ (and equality holds only when Q is a corner of K), by (8₁)-(8₃); cf. (19), (26) and (27) below. Also,

$$\int_K G(P, Q) dK = 1$$

for every P in S ; cf. the case $(f_1, f_2, f_3) \equiv (1, 1, 1)$ above. Clearly, the desired maximum principle follows; for $u(P)$ is a weighted average, with non-negative weights and total weight 1, of the boundary function $f(Q)$.

Unfortunately, the assumptions of the uniqueness theorem (II) do not correspond to the physical situation. Correspondingly, since Hadamard's critique of the various types of "well-set" problems on partial differential equations, the following observations seem to be generally known (cf., e. g. [1], pp. 365-366, where the objection is only implicit).

For the *parabolic* differential equation in the plane, the requirement of two-dimensional continuity, in case of a "well-set" boundary value problem, is quite artificial, since there is only *one* sheaf of characteristic curves.

The physical content of this objection is readily realized. In fact, since u , t , x in (1) denote temperature, time and position on a rod, respectively, the data of the boundary value problem, a problem to which the above statements, (II) and (IV), supply theorems of existence and uniqueness, are as follows: The initial temperature $f_3(x)$, at every point of the rod $0 \leq x \leq 1$, and the temperatures, $f_1(t)$ and $f_2(t)$, for every later date ($0 < t \leq 1$), at both ends, $x=0$ and $x=1$, of the rod. But this interpretation makes unnatural indeed the above assumption on the two-dimensional uniform continuity of $u(x, t)$ on S (that is, the two-dimensional continuity on the closure of S). All that is natural to require is the continuity of the temperature distribution (a) *on the rod*, at every *fixed* date t , and (b) *in time*, at every *fixed* point x of the rod. The two "stokers," engaged at $x=0$ and $x=1$, seem to need quite an advance knowledge of analytic functions in the real domain if, while working independently of each other and of the past, they succeed in producing two-dimensional continuity by virtue of the one-dimensional assumptions (a), (b).

On the other hand, it is well-known that the omission of the assumption

of uniform continuity in (II) invalidates the uniqueness statement. This is shown by the classical "source function," defined by

$$(10) \quad \phi(x, t) = xt^{-3/2} \exp(-x^2/4t) \text{ if } t > 0.$$

It is readily verified that (10) is a solution of (1) on S , that the radial limits (6_1) – (6_3) exist and form a continuous function of the position on K , with $f_1(s) \equiv 0$, $f_2(s) = s^{-3/2} \exp(-1/4s)$ and $f_3(s) \equiv 0$. However, (10) is not the solution supplied by (9), since (10) is not uniformly continuous; in fact, it is not bounded on S . This is seen by considering the function (10) on a parabolic arc $t = x^2$, as $x \rightarrow +0$.

Part of these objections can be met by the following existence and uniqueness theorems which, in contrast to (II) and (IV), do not involve the assumption of two-dimensional continuity on the boundary of S :

(V) *Let $f_1(s)$, $f_2(s)$, $f_3(s)$ be bounded, measurable functions on $0 < s < 1$. Then (8_1) – (8_3) exist for (x, t) on S (as Lebesgue integrals) and their sum (9) is a bounded, continuous solution of (1) on the open square S and satisfies (6_1) – (6_3) almost everywhere.*

(VI) *Conversely, if $u(x, t)$ is a bounded, continuous solution of (1) on the open square S , then the limits (6_1) – (6_3) exist almost everywhere and, for (x, t) on S , the function $u(x, t)$ is given by (9) in terms of the functions (8_1) – (8_3) .*

The condition of two-dimensional uniform continuity in (II) can now be relaxed to that of boundedness. In fact, it follows from (VI), (IV) and (II) that if $u(x, t)$ is a bounded, continuous solution of (1) on S and if the limits (6_1) – (6_3) exist for every s on $0 < s < 1$ and form a uniformly continuous function of the position on K , then $u(x, t)$ is uniformly continuous on the open square S .

The corresponding theorem is true for harmonic functions, bounded on C . However, in the case of harmonic functions, the assumption of "bounded on C " can be relaxed to "half-bounded on C ," for example,

$$(11) \quad u(x, t) \geq 0.$$

The analogous statement for solutions of (1) on S is false. This is shown by the classical "source" function (10).

This, the impossibility of relaxing the assumption of "boundedness" to "half-boundedness" in (VI), is quite unfortunate, since "half-boundedness" does not require more than an appeal to the *third* law of thermodynamics, whereas the justification of "boundedness" depends on the outcome of an

argument about the possible fuel supply of the stokers and their observance of the *first* law.

In order to understand this discrepancy between the behavior of certain solutions of (1) and (3), it is necessary to consider more general solutions of (1) than those given by (V). A well-known theorem (cf., e. g. [3]) states that if $u(x, t)$ is a harmonic function on C and satisfies (11), then $u(x, t)$ is representable as the Stieltjes integral of the Poisson kernel with respect to a bounded non-decreasing function. The analogue of such a Poisson-Stieltjes integral for solutions of (1) would be (9), where

$$(12_1) \quad u_1(x, t) = - \int_0^t \theta_x(x, t-s) dF_1(s),$$

$$(12_2) \quad u_2(x, t) = \int_0^t \theta_x(x-1, t-s) dF_2(s)$$

and

$$(12_3) \quad u_3(x, t) = \frac{1}{2} \int_0^1 \{ \theta(x-s, t) - \theta(x+s, t) \} dF_3(s),$$

and $F_1(s)$, $F_2(s)$, $F_3(s)$ are bounded non-decreasing functions on $0 \leq s \leq t < 1$, $0 \leq s \leq t < 1$, $0 < s < 1$, respectively. This direct analogue of the theorem on non-negative harmonic functions fails for solutions of (1); that is, it is not true that if $u(x, t)$ is a continuous solution of (1) on S and satisfies (11), then $u(x, t)$ is given by (9) in terms of (12₁)-(12₃), where $F_1(s)$, $F_2(s)$, $F_3(s)$ are bounded non-decreasing functions on $0 \leq s \leq t < 1$, $0 \leq s \leq t < 1$, $0 \leq s \leq 1$, respectively. This will follow from (VII) and (XI) below. On the other hand, a certain analogue happens to be true; cf. (VIII) below.

Let $F_1(s)$, $F_2(s)$ be defined for $0 \leq s < 1$ and be of bounded variation on every fixed interval $0 \leq s \leq t < 1$,

$$(13_1) \quad \int_0^t |dF_j(s)| < \infty \text{ for } 0 < t < 1 \text{ and } j = 1, 2,$$

and let $F_3(s)$ be defined for $0 < s < 1$ and of bounded variation on every fixed interval $0 < x_0 \leq s \leq x_1 < 1$; finally, let

$$(13_2) \quad \int_0^1 s(1-s) |dF_3(s)| < \infty.$$

The correct analogues of the theorems on non-negative harmonic functions are given by the following theorems:

(VII) Let $F_1(s)$, $F_2(s)$, $F_3(s)$ be non-decreasing functions on $0 \leq s < 1$, $0 \leq s < 1$, $0 < s < 1$, respectively, satisfying (13₁)-(13₃). Then the sum (9) of the integrals (12₁)-(12₃) represents a non-negative continuous solution of (1) on S .

(VIII) Conversely, if $u(x, t)$ is a non-negative continuous solution of (1) on S , then there exist non-decreasing functions $F_1(s)$, $F_2(s)$, $F_3(s)$ on $0 \leq s < 1$, $0 \leq s < 1$, $0 < s < 1$, respectively, which satisfy (13₁)-(13₃), and $u(x, t)$ is given by (9) in terms of the functions (12₁)-(12₃).

It will be clear from the proof that the integrals in (12₁)-(12₃) are ordinary Riemann-Stieltjes integrals (if $\theta_x(x, 0)$ is defined to be 0, cf. (17)), and that the integral (12₃) is an absolutely convergent improper Riemann-Stieltjes integral.

There is also an analogue for the theorem of Ostrowski [6] on harmonic functions. This theorem states that if $u(x, t)$ is a continuous solution of (3) on C , then $u(x, t)$ can be represented as the Stieltjes integral of the Poisson kernel with respect to a function of bounded variation if and only if

$$\limsup_{r \rightarrow 1-0} \int_0^{2\pi} |u(r \cos \theta, r \sin \theta)| d\theta < \infty.$$

For a corresponding theorem for solutions of (2), one might expect that the Poisson-Stieltjes integral should be replaced by the sum of (12₁)-(12₃), where $F_1(s)$, $F_2(s)$, $F_3(s)$ are of bounded variation on $0 \leq s \leq t < 1$, $0 \leq s \leq t < 1$ and $0 \leq s \leq 1$, respectively; and that the condition in the last formula line should be replaced by

$$(14_1) \quad \limsup_{x \rightarrow +0} \int_0^t |u(x, s)| ds < \infty; \quad (14_2) \quad \limsup_{x \rightarrow 1-0} \int_0^t |u(x, s)| ds < \infty$$

for $0 < t < 1$, and by

$$(14_3^*) \quad \limsup_{t \rightarrow +0} \int_0^1 |u(s, t)| ds < \infty.$$

Actually, this direct analogue of the theorem of Ostrowski is false; cf. the paragraph following (XI) and its Corollary. In the correct analogue, the true conditions are (14₁)-(14₃), where

$$(14_3) \quad \limsup_{t \rightarrow 0} \int_0^1 s(1-s) |u(s, t)| ds < \infty$$

must replace (14_3^*) , and the more general conditions (13_1) – (13_3) must replace the conditions that $F_1(s)$, $F_2(s)$, $F_3(s)$ are of bounded variation on the respective domains of integration in (12_1) , (12_2) , (12_3) .

The situation is as follows: If $u(x, t)$ is a continuous solution of (1) on S and satisfies (14_1) – (14_2) and (14_3^*) , then $u(x, t)$ has a representation of the form (9) in terms of (12_1) – (12_3) , where $F_1(s)$, $F_2(s)$, $F_3(s)$ are of bounded variation on the corresponding domains of integration. This will be clear from the proof of (X). However, the converse is false; that is, if $u(x, t)$ is given by (9) in terms of (12_1) – (12_3) , where $F_1(s)$, $F_2(s)$, $F_3(s)$ are of bounded variation on the domains of integration in (12_1) – (12_3) , then (14_1) – (14_3) hold, but (14_3^*) can fail to hold. As will be proved below in the paragraph following the italicized assertion (XI) and its Corollary, an example to this effect is furnished by the function (10).

(IX) *Let $F_1(s)$, $F_2(s)$, $F_3(s)$ be functions defined on $0 \leq s < 1$, $0 \leq s < 1$, $0 < s < 1$, respectively, and satisfying (13_1) – (13_3) . Then the sum (9) of the integrals (12_1) – (12_3) represents a continuous solution of (1) on S and satisfies (14_1) – (14_3) .*

(X) *Conversely, if $u(x, t)$ is a continuous solution of (1) on S and satisfies (14_1) – (14_3) , then there exist functions $F_1(s)$, $F_2(s)$, $F_3(s)$ defined on $0 \leq s < 1$, $0 \leq s < 1$, $0 < s < 1$, respectively, satisfying (13_1) – (13_3) , and $u(x, t)$ is given by (9) in terms of the functions (12_1) – (12_3) .*

As in the case of harmonic functions, there is a uniqueness theorem corresponding to the existence theorems (VII)–(VIII), (IX)–(X). This uniqueness theorem will be deduced from inversion formulae.

(XI) *Let $F_j(s)$, where $j = 1, 2, 3$, satisfy the conditions of (IX) and let $u(x, t)$ denote the sum (9) of the functions (12_1) – (12_3) . Then, for $0 < t < 1$,*

$$(15_1) \quad \lim_{x \rightarrow 0} \int_0^t u(x, s) ds = F_1(t-0) - F_1(0),$$

$$(15_2) \quad \lim_{x \rightarrow 1+0} \int_0^1 u(x, s) ds = F_2(t-0) - F_2(0)$$

and, for $0 < x < y < 1$,

$$(15_3) \quad \lim_{t \rightarrow +0} \int_x^y u(x, s) ds = \frac{1}{2} \{F_3(y+0) + F_3(y-0)\} \\ - \frac{1}{2} \{F_3(x+0) + F_3(x-0)\}$$

(the integrals in (15₁)-(15₃) exist as Lebesgue integrals).

These inversion formulae imply the following

COROLLARY. If $u(x, t)$ is a continuous solution of (1) on S satisfying (14₁)-(14₃), then, in its representation (9) in terms of (12₁)-(12₃), the functions $F_1(s)$, $F_2(s)$, $F_3(s)$ are uniquely determined, up to additive constants, on their continuity points of the open interval $0 < s < 1$; also, the jumps $F_1(+0) - F_1(0)$, $F_2(+0) - F_2(0)$ are uniquely determined.

It will now be shown that the function $u = \phi(x, t)$ in (10) has a representation (9) in terms of (12₁)-(12₃), where $F_1(s)$, $F_2(s)$, $F_3(s)$ are of bounded variation on $0 \leq s \leq 1$, although (14₃^{*}) is false. The integral in (14₁) and/or (14₂) can be written as the integral of $\exp(-s^2)$ from $s = \frac{1}{2}x/t^{\frac{1}{2}}$ to $s = \infty$ if $\frac{1}{2}x/s^{\frac{1}{2}}$ is introduced as the new integration variable; so that (14₁)-(14₂) hold for $0 < t < 1$. The corresponding integral in (14₃) is, up to a constant factor,

$$\int_0^1 s^2(1-s)t^{-3/2} \exp(-s^2/4t) ds < \int_0^1 s^2t^{-3/2} \exp(-s^2/4t) ds.$$

If $\frac{1}{2}s/t^{\frac{1}{2}}$ is introduced as a new integration variable, the last integral becomes the integral of $4s^2 \exp(-s^2)$ from $s = 0$ to $s = \frac{1}{2}t^{-\frac{1}{2}}$, which is majorized by

$$4 \int_0^\infty s^2 \exp(-s^2) ds < \infty;$$

so that (14₃) holds. Consequently, (X) implies that (10) has a representation on S of the form (9) in terms of (12₁)-(12₃), where $F_1(s)$, $F_2(s)$, $F_3(s)$ satisfy (13₁)-(13₃). But the inversion formula (15₃) shows that $F_3(s) \equiv \text{const.}$; in fact, the integral in (15₃) is a constant multiple of

$$\int_x^y st^{-3/2} \exp(-s^2/4t) ds = 2t^{-\frac{1}{2}} \{\exp(-x^2/4t) - \exp(-y^2/4t)\},$$

which tends to 0, as $t \rightarrow 0$, for $0 < x < y$. Thus, not only do the functions

$F_1(s)$, $F_2(s)$, $F_3(s)$ satisfy (13₁)-(13₃) but they are of bounded variation, on $0 \leq s \leq 1$, as well. Nevertheless, (14₃*) fails to hold, since, to a constant factor, the integral in (14₃*) is

$$\int_0^1 st^{-3/2} \exp(-s^2/4t) ds = 2t^{-1}\{1 - \exp(-1/4t)\}$$

and therefore tends to ∞ , as $t \rightarrow +0$.

It may be mentioned that in the representation of the function (10) as the sum of (12₁)-(12₃), the functions F_1 , F_2 , F_3 , up to additive constants, are given by

$$F_1(0) = 0, \quad F_1(s) = \frac{1}{2}\pi^{-1} \text{ if } s > 0;$$

$$F_2(s) = \frac{1}{2}\pi^{-1} \int_0^s r^{-3/2} \exp(-1/4r) dr; \quad F_3(s) \equiv 0.$$

Still another theorem, analogous to one for harmonic functions, deals with the existence of one-dimensional boundary limits:

(XII) Let $F_1(s)$, $F_2(s)$, $F_3(s)$ satisfy the conditions of (IX) and let $u(x, t)$ denote the sum (9) of the functions (12₁)-(12₃). Then, for $j = 1, 2, 3$, the limit (6_j) exists at every s on $0 < s < 1$ at which the derivative $F_j'(s)$ exists; in fact, for such an s -value, $f_j(s) = F_j'(s)$. (In particular, the limits (6₁)-(6₃) exist almost everywhere.)

It may be remarked that if $j = 1, 2$ or 3 , then (12_j) reduces to (8_j) if and only if $F_j(s)$ is absolutely continuous. Consequently, a solution $u(x, t)$ of (1) on S can possess limits (6₁)-(6₃), which are bounded, without $u(x, t)$ being bounded on S and, therefore, without possessing a representation as the sum of integrals (8₁)-(8₃). In fact, if $u(x, t)$ is representable as the sum of integrals (12₁)-(12₃), then $u(x, t)$ is bounded on S if and only if $F_1(s)$, $F_2(s)$, $F_3(s)$ are absolutely continuous on the domains of integration and possess bounded derivatives almost everywhere.

Before proving the assertions (V)-(XII), it will be convenient first to prove a part of (IX).

First part of the proof of (IX). It is readily verified from the identity

$$(16) \quad \theta(x, t) = \pi^{-1} \sum_{k=-\infty}^{+\infty} t^{-1} \exp(-(x+2k)^2/4t), \quad t > 0,$$

supplied by the linear transformation of the ϑ_3 -function, that

$$(17) \quad \theta_x(x, t) \rightarrow 0 \text{ as } t \rightarrow +0, \quad (0 < x < 2).$$

In fact, the series (16) and its derived series are uniformly convergent for $0 < x_0 \leq x \leq x_1 < 2$ and $0 < t \leq t_1 < \infty$; while, on this x -range, each term tends uniformly to 0 as $t \rightarrow +0$ (cf. the formulae following (53₁) below). Hence, (17) holds uniformly on such an x -interval. Clearly, (17) implies that, if $F_1(s)$, $F_2(s)$ are of bounded variation on $0 \leq s \leq t$, then the integrals in (12₁), (12₂) exist, as Riemann-Stieltjes integrals, for every fixed (x, t) in S . Furthermore, the remark concerning the uniformity of (17) shows that (12₁), (12₂) are continuous functions on S .

That (12₁) is a solution of (1) on S can be proved directly. However, in order to simplify matters in subsequent proofs, a somewhat indirect proof will be employed. Put

$$(18) \quad U_1(x, t) = \int_0^t u_1(x, r) dr.$$

Since, formally,

$$\int_0^t u_1(x, r) dr = - \int_0^t \int_0^r \theta_x(x, r-s) dF_1(s) dr,$$

the existence of the integral in (18) will be proved if it is shown that the iterated integral

$$- \int_0^t \int_s^t \theta_x(x, r-s) dr dF_1(s) = - \int_0^t \int_0^{t-s} \theta_x(x, r) dr dF_1(s)$$

converges absolutely. To this end, since $F_1(s)$ is of bounded variation on $0 \leq s \leq t$, it can be supposed that $F_1(s)$ is non-decreasing (otherwise $F_1(s)$ is written as the difference of two non-decreasing functions, and each of the corresponding integrals is treated separately). Also, since

$$(19) \quad -\theta_x(x, t) > 0 \text{ for } 0 < x < 1 \text{ and } 0 < t < \infty$$

(cf., e. g., [8], p. 410), the last iterated integral can be treated without the insertion of absolute value signs. But (17) and an integration by parts show that the integral exists and is

$$(20) \quad U_1(x, t) = - \int_0^t \theta_x(x, t-s) (F_1(s) - F_1(0)) ds.$$

Thus, (18) can be written in the form (8₁), where $f_1(s) = F_1(s) - F_1(0)$ is Riemann integrable for $0 \leq s \leq t < 1$ (for t fixed). Consequently, by

standard theorems (cf., e. g., [1], p. 355), (18) is a continuous solution of (1) on S . Hence, by (I), the function (18) is of class C^∞ on S . Since (12₁) is continuous on S , the function $u_1(x, t) = \partial U_1(x, t)/\partial t$ is a continuous solution of (1) on S .

Similarly,

$$(21) \quad U_2(x, t) = \int_0^t u_2(x, r) dr$$

exists as a Lebesgue integral. Furthermore,

$$(22) \quad U_2(x, t) = \int_0^t \theta_x(x-1, t-s) (F_2(s) - F_2(0)) ds$$

is of class C^∞ and a solution of (1) on S . Hence, the function (12₂), which is $\partial U_2/\partial t$, is a continuous solution of (1) on S .

In order to see that (12₃) exists and is a continuous solution of (1) on S , note that (7) implies that

$$(23) \quad \frac{1}{2} \{ \theta(x-s, t) - \theta(x+s, t) \} = 2 \sum_{k=1}^{\infty} \exp(-k^2 \pi^2 t) \sin k\pi x \sin k\pi s.$$

Hence, for a fixed (x, t) in S , this function vanishes as s (or $1-s$) at $s=0$ (or $s=1$). Consequently, if $F_3(s)$ is defined on $0 < s < 1$ and satisfies (13₃), the integral in (12₃) converges absolutely as an improper Riemann-Stieltjes integral. In fact, since (13₃) implies that

$$(24) \quad \int_0^1 |\sin k\pi s| |dF_3(s)| \text{ exists and is } O(k), \text{ as } k \rightarrow \infty,$$

the series (23) can be integrated term-by-term. Clearly,

$$(25) \quad u_3(x, t) = 2 \sum_{k=1}^{\infty} \exp(-k^2 \pi^2 t) \sin k\pi x \int_0^1 \sin \pi s dF_3(s)$$

is defined and continuous on S . Since $\exp(-k^2 \pi^2 t) \sin k\pi x$, where $k=1, 2, \dots$, is a solution of (1), and since (25) can be differentiated (repeatedly) term-by-term at any point (x, t) of S , it follows that (25), that is, (12₃), is a continuous solution of (1) on S .

In order to complete the proof of (IX), it remains to verify that the sum (9) of (12₁)-(12₃) satisfies (14₁)-(14₃). This will be done below with the aid of Lemma 1.

Proof of (VII). In view of the portion of (IX) just proved, it only remains to verify that the function (12_j) is non-negative when $dF_j \geq 0$. For $j=1$, this is clear from (19). Since, for a fixed t , the function $\theta_x(x, t)$ is even and of period 2 in x , it is seen from (19) that

$$(26) \quad \theta_x(x-1, t) > 0 \text{ for } 0 < x < 1 \text{ and } 0 < t < \infty.$$

Hence (12_2) is non-negative. Finally, the kernel (23) of (12_3) satisfies

$$(27) \quad \theta(x-s, t) - \theta(x+s, t) > 0 \text{ for } 0 < x < 1, 0 < s < 1, t > 0,$$

by (19); so that (12_3) is non-negative.

In order to complete the proof of (IX), a lemma will first be established.

LEMMA 1. *Let $u(x, t)$ be a non-negative, continuous solution of (1) on S . Then, if $0 < x < 1$, the integrals occurring in (14_1) -(14_2) exist as Lebesgue integrals and, if $0 < t < 1$, the integral in (14_3) exists as a Lebesgue integral. Furthermore, (14_1) -(14_3) hold.*

Proof of Lemma 1. Let ϵ, ξ, τ denote numbers satisfying

$$(28) \quad \frac{1}{2} > \epsilon > \xi > 0 \text{ and } \frac{1}{2} > \epsilon > \tau > 0.$$

Since $u(x, t)$ is a continuous solution of (1) on S , the function $u = v(x, t) = v(x, t, \epsilon, \xi, \tau)$ defined by

$$(29) \quad v(x, t) = u(x(1-\epsilon) + \xi, t(1-\epsilon)^2 + \tau)$$

is a solution of (1) on S and is uniformly continuous on the open set S . Hence, by (II) and (IV), the function $u = v(x, t)$ has a representation of the form (9) in terms of (8_1) -(8_3), where the functions $f_j(s) = f_j(s, \epsilon, \xi, \tau)$; $j=1, 2, 3$, are given by

$$(30_1) \quad f_1(s) = u(\xi, s(1-\epsilon)^2 + \tau);$$

$$(30_2) \quad f_2(s) = u(1-\epsilon + \xi, s(1-\epsilon)^2 + \tau)$$

and

$$(30_3) \quad f_3(s) = u(s(1-\epsilon) + \xi, \tau).$$

Since $u(x, t)$ is non-negative, the inequalities (19), (26) and (27) show that each of the corresponding terms of (9) is non-negative and, therefore, none of them exceeds their sum, $v(x, t)$. Hence, if $0 < \delta < t$,

$$(31) \quad v(x, t) \geq - \int_0^t \theta_x(x, t-s) f_1(s) ds \geq - \int_0^{t-\delta} \theta_x(x, t-s) f_1(s) ds.$$

If (x, t) is an arbitrary point of R , there exists a constant $C = C(x, t, \delta)$ satisfying

$$(32) \quad -\theta_x(x, t-s) \geq C > 0 \text{ for } 0 \leq s \leq t-\delta.$$

This can be deduced, for instance, from the Jacobi identity

$$\theta(x, t) = \prod_{n=1}^{\infty} (1 - q^{2n}) (1 + 2q^{2n-1} \cos \pi x + q^{4n-2}),$$

where $q = \exp(-\pi^2 t)$. This identity shows the existence of a constant, depending on δ , for which $\theta(x, t) \geq \text{const.} > 0$ if $0 < \delta < t \leq 1$. Also, logarithmic differentiation of the Jacobi identity with respect to x gives

$$-\theta_x(x, t)/\theta(x, t) = 2\pi \sin \pi x \sum_{n=1}^{\infty} q^{2n-1} (1 + 2q^{2n-1} \cos \pi x + q^{4n-2})^{-1}.$$

Clearly, (31) follows from the last two formula lines in view of the definition of q . (A similar use of the Jacobi identity was made in [8], p. 410.) But (31) and (32) imply that

$$C \int_0^{t-\delta} f_1(s) ds \leq v(x, t).$$

In view of the definitions (30₁) of $f_1(s)$ and (29) of $v(x, t)$, this inequality can be written as

$$C \int_0^{t-\delta} u(\xi, s(1-\epsilon)^2 + \tau) ds \leq u(x(1-\epsilon) + \xi, t(1-\epsilon)^2 + \tau).$$

If δ, ϵ, ξ are fixed, but $\tau \rightarrow 0$ (which is permissible in view of the second set of inequalities in (28)), it follows from Fatou's lemma that

$$C \int_0^{t-\delta} u(\xi, s(1-\epsilon)^2) ds \leq u(x(1-\epsilon) + \xi, t(1-\epsilon)^2),$$

or, by a change of variables,

$$C(1-\epsilon)^{-2} \int_0^t u(\xi, s) ds \leq u(x(1-\epsilon) + \xi, t + \delta(1-\epsilon)^2),$$

where t now stands for $(t-\delta)(1-\epsilon)^2$.

Since $\epsilon, \delta (> 0)$ are arbitrary, it is seen that the integral in (14₁) and/or (14₂) exists. If $\delta, \epsilon (> 0)$ are kept fixed, while $\xi \rightarrow 0$, the last formula line implies

$$\limsup_{\xi \rightarrow 0} \int_0^{\xi} u(\xi, s) ds \leq (1 - \epsilon)^2 u(x(1 - \epsilon), t + \delta(1 - \epsilon)^2) / C < \infty,$$

that is, that (14₁) holds. In a similar manner, it can be shown that (14₂) holds.

The proof of the existence of the integral in (14₃) requires a slight modification of the procedure above. By the argument leading to (31),

$$(33) \quad v(x, t) \geq \frac{1}{2} \int_0^1 \{\theta(x - s, t) - \theta(x + s, t)\} f_s(s) ds.$$

For a fixed $t > 0$, the function $\theta(x, t)$ is even, of period 2 in x , and strictly increasing for $0 < x < 1$ (cf. (19)). Hence to any $\delta > 0$, there corresponds a constant $C = C(\delta)$ such that

$$\frac{1}{2} \{\theta(x - s, t) - \theta(x + s, t)\} \geq C > 0 \text{ for } \delta \leq s \leq 1 - \delta.$$

In addition,

$$(34) \quad \frac{1}{2} \{\theta(x - s, t) - \theta(x + s, t)\} / s \rightarrow -\theta_x(x, t) \text{ as } s \rightarrow 0$$

and

$$(35) \quad \frac{1}{2} \{\theta(x - s, t) - \theta(x + s, t)\} / (1 - s) \rightarrow \theta_x(x + 1, t) \text{ as } s \rightarrow 1.$$

Consequently,

$$\frac{1}{2} \{\theta(x - s, t) - \theta(x + s, t)\} \geq C^* s(1 - s) \text{ for } 0 \leq s \leq 1$$

for some positive constant C^* , depending on the point (x, t) of S . Thus, (33) shows that

$$C^* \int_0^1 s(1 - s) f_s(s) ds \leq v(x, t).$$

In view of the definitions, (30₃) of $f_s(s)$ and (29) of $v(x, t)$,

$$C^* \int_0^1 s(1 - s) u(s(1 - \epsilon) + \xi, \tau) ds \leq u(x(1 - \epsilon) + \xi, t(1 - \epsilon)^2 + \tau).$$

Keeping ϵ, τ fixed, let $\xi \rightarrow 0$. Then, by Fatou's lemma,

$$C^* \int_0^1 s(1 - s) u((s(1 - \epsilon), \tau) ds \leq u(x(1 - \epsilon), t(1 - \epsilon)^2 + \tau).$$

A change of variables shows that, if $\epsilon > 0$ is sufficiently small,

$$C^*(1-\epsilon)^{-2} \int_0^{\frac{1}{2}} s(1-s)u(s, \tau) ds \leq u(x(1-\epsilon), t(1-\epsilon)^2 + \tau).$$

A repetition of the above arguments, with x replaced by $1-x$, leads to

$$C^{**}(1-\epsilon)^{-2} \int_{\frac{1}{2}}^1 s(1-s)u(s, \tau) ds \leq u((1-x)(1-\epsilon), t(1-\epsilon)^2 + \tau),$$

where C^{**} is a positive constant depending on the point $(1-x, t)$ of S .

These inequalities imply that the integral in (14_3) exists as a Lebesgue integral. By keeping $\epsilon > 0$ fixed and letting $\tau \rightarrow 0$ in the last two formula lines, the upper limit relation (14_3) follows. This completes the proof of Lemma 1.

Completion of the proof of (IX). It is sufficient to verify that each of the functions $u = u_j(x, t)$ given by (12_j) , where $j = 1, 2, 3$, satisfies (14_1) - (14_3) . For, if (14_1) - (14_3) holds for two or more functions, it clearly holds for their sum. In (12_1) , it can be supposed that $F_1(s)$, a function of bounded variation in $0 \leq s \leq t < 1$, is non-decreasing (for otherwise $F_1(s)$ is written as the difference of two non-decreasing functions, and each of the two corresponding functions (12_1) is treated separately). But if $F_1(s)$ is non-decreasing, then (VII), proved above, implies that (12_1) is non-negative. Then (14_1) - (14_3) follow from Lemma 1. In a similar manner, it is shown that $u = u_2(x, t)$, given by (12_2) , satisfies (14_1) - (14_3) .

If a function $F_3(s)$, defined on $0 < s < 1$ and satisfying (13_3) , can be written as the difference of two non-decreasing functions each of which satisfies (13_3) , then it follows, in the same way as above, that $u = u_3(x, t)$, given by (12_3) , satisfies (14_1) - (14_3) . Finally, it is clear that the usual device used for decomposing functions of bounded variation,

$$F_3(s) = \int_{\frac{1}{2}}^s |dF_3(s)| - \left(\int_{\frac{1}{2}}^s |dF_3(s)| - F_3(s) \right), \quad (0 < s < 1),$$

yields the desired decomposition. This completes the proof of (IX).

Proof of (VIII). It will be shown that (VIII) follows from Lemma 1 and from the assertions (X) and (XI), to be proved below. In fact, Lemma 1 and (X) imply that, if $u(x, t)$ is a non-negative, continuous solution of (1) on S , then there exist functions $F_1(s)$, $F_2(s)$, $F_3(s)$ satisfying (13_1) - (13_3)

and having the property that $u(x, t)$ possesses a representation (9) in terms of (12_1) -(12_3). Finally, the inversion formulae in (XI) and the non-negativity of $u(x, t)$ imply that the functions $F_1(s)$, $F_2(s)$, $F_3(s)$ are non-decreasing.

Proof of (X). Let $u(x, t)$ be a continuous solution of (1) on S satisfying (14_1) -(14_3). Let $0 < \epsilon < \frac{1}{2}$ and put

$$(36) \quad u^\epsilon(x, t) = u(x(1-2\epsilon) + \epsilon, t(1-2\epsilon)^2 + \epsilon).$$

Then $u^\epsilon(x, t)$ is uniformly continuous on S and represents a solution of (1) on S . By (II) and (IV), the function $u = u^\epsilon(x, t)$ possesses a representation (9) in terms of functions (8_1) -(8_3). The Riemann integrals (8_1) -(8_3) can be written as Riemann-Stieltjes integrals (12_1) -(12_3). For the case at hand, the function $F_1(s) = F_1^\epsilon(s)$ is given by

$$F_1^\epsilon(s) = \text{const.} + \int_0^s u(\epsilon, r(1-2\epsilon)^2 + \epsilon) dr,$$

or, for a suitable choice of the const., by

$$(37_1) \quad F_1^\epsilon(s) = (1-2\epsilon)^{-2} \int_0^{s(1-2\epsilon)^2 + \epsilon} u(\epsilon, r) dr, \quad (0 < s < 1).$$

Similarly,

$$(37_2) \quad F_2^\epsilon(s) = (1-2\epsilon)^{-2} \int_0^{s(1-2\epsilon)^2 + \epsilon} u(1-\epsilon, r) dr, \quad (0 < s < 1)$$

and

$$(37_3) \quad F_3^\epsilon(s) = (1-2\epsilon)^{-1} \int_{\frac{1}{2}}^{s(1-2\epsilon) + \epsilon} u(r, \epsilon) dr, \quad (0 < s < 1).$$

Hence

$$(9^\epsilon) \quad u^\epsilon(x, t) = u_1^\epsilon(x, t) + u_2^\epsilon(x, t) + u_3^\epsilon(x, t),$$

where

$$(12_1^\epsilon) \quad u_1^\epsilon(x, t) = - \int_0^t \theta_x(x, t-s) dF_1^\epsilon(s),$$

$$(12_2^\epsilon) \quad u_2^\epsilon(x, t) = \int_0^t \theta_x(x-1, t-s) dF_2^\epsilon(s),$$

$$(12_s^\epsilon) \quad u_s^\epsilon(x, t) = \frac{1}{2} \int_0^1 \{\theta_s(x-s, t) - \theta_s(x+s, t)\} dF_s^\epsilon(s).$$

Clearly, the assumptions (14₁)-(14₃) imply the existence of the functions (37₁)-(37₃). Also, (14₁) implies that, for any t on $0 < t < 1$, the functions $F_1^\epsilon(s)$ are uniformly bounded and of uniform bounded variation for $0 \leq s \leq t$. Helly's selection theorem implies therefore the existence of a function $F_1(s)$, $0 \leq s < 1$, which is of bounded variation on every interval $0 \leq s \leq t < 1$, and the existence of a sequence of positive numbers $\epsilon_1^1, \epsilon_2^1, \dots$ satisfying

$$\lim_{n \rightarrow \infty} \epsilon_n^1 = 0 \text{ and } \lim_{n \rightarrow \infty} F_1^{\epsilon_n^1}(s) = F_1(s), \quad (\epsilon = \epsilon_n^1),$$

where $0 < s < 1$. In addition, (17) and Helly's term-by-term integration for Riemann-Stieltjes integrals imply that the function (12₁^ε) where $\epsilon = \epsilon_n^1$, tends to (12₁) as $n \rightarrow \infty$, for every (x, t) in S .

The arguments applied to the family of functions $F_2^\epsilon(s)$, where $\epsilon = \epsilon_n^1$, also show the existence of a function $F_2(s)$, of bounded variation on every interval $0 \leq s \leq t < 1$, and of a subsequence $\epsilon_1^2, \epsilon_2^2, \dots$ of the sequence $\epsilon_1^1, \epsilon_2^1, \dots$, having the property that (12₂^ε), where $\epsilon = \epsilon_n^2$, tends to (12₂) as $n \rightarrow \infty$, for every (x, t) in S .

Finally, by considering the family of functions $F_3^\epsilon(s)$, where $\epsilon = \epsilon_n^2$, it is seen that there exist a function $F_3(s)$, $0 < s < 1$, of bounded variation on $\delta \leq s \leq 1 - \delta$ for every small positive δ , and a subsequence $\epsilon_1^3, \epsilon_2^3, \dots$ of the sequence $\epsilon_1^2, \epsilon_2^2, \dots$ in such a way that

$$\lim_{n \rightarrow \infty} F_3^{\epsilon_n^3}(s) = F_3(s) \quad (\epsilon = \epsilon_n^3)$$

holds on $0 < s < 1$. Furthermore, (14₃) implies that

$$\limsup_{\epsilon \rightarrow 0} \int_0^1 s(1-s) |dF_s^\epsilon(s)| < \infty, \text{ hence } \int_0^1 s(1-s) |dF_s(s)| < \infty.$$

Hence, if $G^\epsilon(s)$ and $G(s)$ are defined by $dG^\epsilon(s) = s(1-s)dF_s^\epsilon(s)$ and $dG(s) = s(1-s)dF_s(s)$, respectively, these functions are of uniformly bounded variation. If the kernel in the integral in (12₃^ε) is divided by $s(1-s)$, the relations (34) and (35) and an application of Helly's term-by-term integration theorem show that (12₃^ε) tends to (12₃) as $n \rightarrow \infty$ (where $\epsilon = \epsilon_n^3$).

Consequently, if $\epsilon = \epsilon_n^3$ in (9^ε) and $n \rightarrow \infty$, the right side of (9^ε) tends to (9), where u_1, u_2, u_3 are given by (12₁), (12₂), (12₃), respectively. On the other hand, the left side of (9^ε) tends to $u(x, t)$ as $\epsilon \rightarrow 0$, by virtue of

(36) and the continuity of $u(x, t)$ on S . This completes the proof of the representation (9) for the given function $u(x, t)$ in terms of integrals of the form (12₁)-(12₃).

For the proof of (XI) and (XII), it will be convenient to have the following localization theorem:

LEMMA 2. Let $F_3(s)$ be defined on $0 < s < 1$, be of bounded variation on every closed interval of $0 < s < 1$, and such as to satisfy (13₃). Let $0 < \delta < 1$ and $0 < \delta < x < 1 - \delta$. Then

$$(38) \quad \lim_{t \rightarrow +0} \frac{1}{2} \int_0^\delta \{\theta(x-s, t) - \theta(x+s, t)\} dF_3(s) = 0$$

and

$$(39) \quad \lim_{t \rightarrow +0} \frac{1}{2} \int_{1-\delta}^1 \{\theta(x-s, t) - \theta(x+s, t)\} dF_3(s) = 0,$$

and both of these limit relations are uniform with respect to x and δ if $0 < \text{const.} \leq x - \delta$ and $1 - x - \delta \leq \text{const.} > 0$.

Proof of Lemma 2. If $s > 0$, the mean-value theorem of differential calculus gives

$$\frac{1}{2} \{\theta(x-s, t) - \theta(x+s, t)\} / s = -\theta_x(x + \xi s, t),$$

where $|\xi| < 1$ and $\xi = \xi(x, s, t)$. Since (17) holds uniformly on any interval $0 < x_0 \leq x \leq x_1 < 2$, it follows that, if $\epsilon > 0$ is given,

$$\frac{1}{2} |\theta(x-s, t) - \theta(x+s, t)| / s < \epsilon$$

holds for $0 < s \leq \delta$ and for sufficiently small $t > 0$. But then the integral

in (38) is majorized by $\epsilon \int_0^\delta s |dF_3(s)|$ for small $t > 0$. In view of (13₃),

this proves (38). The relation (39) is proved in the same way; cf. (35).

The last part of Lemma 2, concerning the uniformity of (38) and (39), is clear from the above proof.

LEMMA 3. Let $F_3(s)$ be defined on $0 < s < 1$, be of bounded variation on every closed interval on $0 < s < 1$, and such as to satisfy (13₃). Then the series

$$(40) \quad 2 \sum_{k=1}^{\infty} (k\pi)^{-2} \sin k\pi x \int_0^1 \sin k\pi s dF_3(s) = G(x)$$

is uniformly convergent for $0 \leq x \leq 1$; in particular

$$(41) \quad \lim_{x \rightarrow 0} G(x) = 0 \text{ and } \lim_{x \rightarrow 1} G(x) = 0.$$

Proof. First, it will be shown that (13₃) implies

$$(42) \quad s \int_s^{\frac{1}{2}} |dF_3(r)| \rightarrow 0 \text{ and } s \int_{\frac{1}{2}}^{1-s} |dF_3(r)| \rightarrow 0, \quad (s \rightarrow 0),$$

and

$$(43) \quad \int_0^{\frac{1}{2}} \int_s^{\frac{1}{2}} |dF_3(r)| ds < \infty \text{ and } \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^s |dF_3(r)| ds < \infty.$$

Let $\epsilon > 0$ be given and let $\delta = \delta_\epsilon > 0$ be chosen so small that

$$\epsilon > \int_0^\delta r |dF_3(r)| \geq \int_s^\delta r |dF_3(r)| \geq s \int_s^\delta |dF_3(r)|.$$

Since, if $\delta > 0$ fixed, $s \int_{\delta}^{\frac{1}{2}} |dF_3(r)| \rightarrow 0$ as $s \rightarrow 0$, the first part of (42)

follows, $\epsilon > 0$ being arbitrary. The second part of (42) is proved in the same way.

By Fubini's theorem on iterated integrals, the integrals in (43) are

$$\int_0^{\frac{1}{2}} s |dF_3(s)| \text{ and } \int_{\frac{1}{2}}^1 (1-s) |dF_3(s)|,$$

respectively, and are finite by (13₃).

Let $G(x)$ denote, for a moment, the function defined on $0 \leq x < \frac{1}{2}$ by

$$(44_1) \quad G(x) = \int_0^x \int_s^{\frac{1}{2}} dF_3(r) ds + \alpha x$$

and on $\frac{1}{2} < x \leq 1$ by

$$(44_2) \quad G(x) = \int_x^1 \int_{\frac{1}{2}}^s dF_3(r) ds + \alpha x + \beta,$$

In view of (43), these definitions are meaningful. If the constants α and β are chosen to be

$$\alpha = -\beta = \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^s dF_3(r) ds - \int_0^{\frac{1}{2}} \int_s^{\frac{1}{2}} dF_3(r) ds,$$

it is seen that $G(\frac{1}{2}-0) = G(\frac{1}{2}+0)$; so that $G(x)$ can be defined at $x = \frac{1}{2}$ so as to be continuous there. The above choice of α, β assures that

$$(45) \quad G(0) = G(1) = 0.$$

The relations (43) show that $G(x)$ is absolutely continuous for $0 \leq x \leq 1$ and that

$$(46) \quad G'(x) \text{ is } \int_x^{\frac{1}{2}} dF_3(r) + \alpha \text{ or } -\int_{\frac{1}{2}}^x dF_3(r) + \alpha$$

according as $0 < x \leq \frac{1}{2}$ or $\frac{1}{2} \leq x < 1$. The k -th Fourier sine coefficient of $G(x)$ on $0 < x < 1$ is, by (45),

$$2 \int_0^\pi G(x) \sin k\pi x \, dx = 2(k\pi)^{-1} \int_0^\pi G'(x) \cos k\pi x \, dx,$$

the last integral existing in virtue of (43). Since (42) implies that $G'(x) \sin k\pi x \rightarrow 0$ as $x \rightarrow +0$ or $x \rightarrow 1-0$, another integration by parts and (46) show that the expression on the right in the last formula line is

$$2(k\pi)^{-2} \int_0^1 \sin k\pi x \, dF_3(x).$$

Hence the series in (40) is the Fourier sine series for $G(x)$. Since $G(x)$ is of bounded variation, continuous on $0 \leq x \leq 1$, and vanishes at $x=0$ and $x=1$, the uniform convergence of the series (40) follows. This proves Lemma 3.

Proof of (XI). Let $u(x, t)$ be a continuous solution of (1) on S , representable in the form (9) in terms of functions (12₁)-(12₃). It will first be shown that, for $0 < t < 1$,

$$(47_1) \quad \lim_{x \rightarrow +0} \int_0^t u_1(x, s) \, ds = F_1(t-0) - F_1(0).$$

In the proof of (47₁), use will be made of (18) and (20), as follows: The function $U_1(x, t)$ has a representation of the form (9) in terms of functions of the form (8₁)-(8₃), where $f_1(s), f_2(s), f_3(s)$ are $F_1(s) - F(0), 0, 0$, respectively. Hence, by standard theorems (cf. [1], p. 355), (47₁) holds almost everywhere. Actually, the limit in (6₁) exists and is $f_1(s-0)$ for every point s at which $f_1(s)$ is continuous from the left (cf. [1], p. 355).

(This can be deduced from the standard treatment of Lebesgue for "singular integrals"; cf. the proof of (XII). Incidentally, (47₁) follows from (XII) itself.) Hence, (47₁) holds for every t on $0 < t < 1$.

These arguments also show that

$$(47_2) \quad \lim_{x \rightarrow 1-0} \int_0^t u_1(x, s) ds = 0, \quad (0 < t < 1).$$

The limit relation

$$(47_3) \quad \lim_{t \rightarrow +0} \int_0^t u_1(x, s) ds = 0, \quad (0 < x < 1),$$

is a consequence of the existence of the integral (47₃) as a Lebesgue integral.

In the same way as (18), (20) lead to (47₁) and (47₂), it is seen that (21), (22) lead to

$$(48_1) \quad \lim_{x \rightarrow +0} \int_0^t u_2(x, s) ds = 0, \quad (0 < t < 1),$$

and

$$(48_2) \quad \lim_{x \rightarrow 1-0} \int_0^t u_2(x, s) ds = F_2(t-0) - F_2(0) \quad (0 < t < 1).$$

In addition, the proof of (47₃) shows that

$$(48_3) \quad \lim_{t \rightarrow +0} \int_0^t u_2(x, s) ds = 0, \quad (0 < x < 1).$$

Finally, it will be shown that

$$(49_1) \quad \lim_{x \rightarrow +0} \int_0^t u_3(x, s) ds = 0, \quad (0 < t < 1)$$

and

$$(49_2) \quad \lim_{x \rightarrow 1-0} \int_0^t u_3(x, s) ds = 0, \quad (0 < t < 1)$$

and that, for $0 < x < y < 1$,

$$(49_3) \quad \lim_{t \rightarrow +0} \int_x^y u_3(s, t) ds = \frac{1}{2} \{F_3(y+0) + F_3(y-0)\} \\ - \frac{1}{2} \{F_3(x+0) + F_3(x-0)\}.$$

If $0 < \epsilon < t$, it is clear from (25) that

$$\int_{\epsilon}^t u_3(x, s) ds = 2 \sum_{k=1}^{\infty} (k\pi)^{-2} \{ \exp(-k^2\pi^2\epsilon) - \exp(-k^2\pi^2 t) \} \sin k\pi x \int_0^1 \sin k\pi s dF_3(s).$$

Since the integral in (49₁) and/or (49₂) exists, the convergence of (40) and the regularity of Abel's summation method show that the last equation becomes

$$\int_0^t u_3(x, s) ds = 2 \sum_{k=1}^{\infty} (k\pi)^{-2} \{ 1 - \exp(-k^2\pi^2 t) \} \sin k\pi x \int_0^1 \sin k\pi s dF_3(s),$$

as $\epsilon \rightarrow +0$. (Strictly speaking, $\epsilon \rightarrow +0$ is not Abel's summation method; however, if 0 terms are inserted to take care of terms $\exp(-k^2\pi^2\epsilon)$, for k not a square, Abel's theorem becomes applicable.) For a fixed $t > 0$, the last series is uniformly convergent for $0 \leq x \leq 1$. Hence (49₁), (49₂) follow.

In order to prove (49₃), let δ be so chosen that $0 < \delta < x < y < 1 - \delta$. Then the uniformity statement concerning (38), (39) in Lemma 2 implies that the expression on the left of (49₃) is the limit of

$$\int_x^y \frac{1}{2} \int_{\delta}^{1-\delta} \{ \theta(s-r, t) - \theta(s+r, t) \} dF_3(r) ds$$

as $t \rightarrow +0$, provided that this limit exists. For a fixed $t > 0$, the expression in the last formula line is

$$(50) \quad 2 \sum_{k=1}^{\infty} (k\pi)^{-1} \exp(-k^2\pi^2 t) \{ \cos k\pi y - \cos k\pi x \} \int_{\delta}^{1-\delta} \sin k\pi s dF_3(s).$$

Consider a function of bounded variation $H(s)$ defined on $0 \leq s \leq 1$ by letting $dH(s)$ be 0, $dF(s)$ or 0 according as $0 \leq s < \delta$, $\delta < s < 1 - \delta$ or $1 - \delta < s \leq 1$. If the Fourier-Stieltjes sine series of $H(s)$, on $0 \leq s \leq 1$, is integrated from $s = x$ to $s = y$, the result is (50), when $t = 0$. Consequently, when $t = 0$, the series (50) is convergent and its sum is the expression on the right of (49₃), since $0 < \delta < x < y < 1 - \delta$. Finally, (49₃) follows, since $t \rightarrow +0$ in (50) is a regular summation process; cf. the remarks in the proof of (49₁)-(49₂).

The inversion formulae (15₁), (15₂), (15₃) are consequences of (9) and (47₁)-(47₃), (48₁)-(48₃), (49₁)-(49₃), respectively. This completes the proof of (XI).

The proof of (XII) will depend on Lemma 2 and a standard lemma of the Lebesgue-Toeplitz type; cf. [4].

LEMMA 4. For a fixed τ on $0 < \tau \leq 1$, let $\psi(\sigma, \tau)$ be a continuous function of bounded variation on the interval $|\sigma| \leq l \leq \infty$, and possess the following properties:

- (i) $\lim_{\tau \rightarrow +0} \psi(\sigma, \tau) = 0$ for every fixed σ on $0 < |\sigma| \leq l$;
- (ii) $\lim_{\tau \rightarrow +0} \int_{|\sigma| > \delta} |d\sigma\psi(\sigma, \tau)| = 0$ for every fixed δ on $0 < \delta < l$;
- (iii) $\limsup_{\tau \rightarrow +0} \int_{|\sigma| < \eta} |\sigma| |d\sigma\psi(\sigma, \tau)| < \infty$ for some $\eta > 0$;
- (iv) $\lim_{\tau \rightarrow +0} \int_{|\sigma| < \delta} \sigma d\sigma\psi(\sigma, \tau) = -1$ for every fixed δ on $0 < \delta < \eta$.

Let $F(x)$ be defined and bounded for $|x| \leq l$ and of bounded variation on every closed interval on $|x| < l$. Then

$$(51) \quad I(x, \tau) = \int_{\alpha}^{\beta} \psi(x - \sigma, \tau) dF(\sigma),$$

where $\alpha = \max(-l, x - l)$ and $\beta = \min(l, x + l)$, exists and, for every x on $|x| < l$ at which $F(x)$ is differentiable, satisfies

$$(52) \quad \lim_{\tau \rightarrow +0} I(x, \tau) = F'(x).$$

Proof of Lemma 4. The $dF(\sigma)$ in (51) can be replaced by $d\{F(\sigma) - F(x)\}$ for any fixed x . Let $x - \sigma$ be introduced as a new integration variable in (51). Property (i) and an application of integration by parts to (51) show that $I(x, t)$ exists and that

$$I(x, \tau) = \int_{x-\beta}^{x-\alpha} \{F(x - \sigma) - F(x)\} d\sigma\psi(\sigma, \tau) + o(1)$$

as $\tau \rightarrow +0$. If $|x| < l$, then, for some δ , where $0 < \delta < \eta$,

$$I(x, \tau) = - \int_{|\sigma| < \delta} \{F(x) - F(x - \sigma)\} d\sigma\psi(\sigma, \tau) + o(1)$$

as $\tau \rightarrow +0$, by virtue of property (ii) and the boundedness of $F(\sigma)$. If $F'(x)$ exists, property (iv) implies that

$$I(x, \tau) - F'(x) = \int_{|\sigma| < \delta} \{F'(x) - (F(x) - F(x - \sigma))/\sigma\} \sigma d\sigma\psi(\sigma, \tau) + o(1)$$

as $\tau \rightarrow +0$. Finally, if $\delta = \delta_\epsilon$ is chosen so small that

$$|F'(x) - (F(x) - F(x - \sigma))/\sigma| < \epsilon \text{ for } |\sigma| < \delta,$$

it is seen that

$$\limsup_{\tau \rightarrow +0} |I(x, \tau) - F'(x)| \leq \epsilon \limsup_{\tau \rightarrow +0} \int_{|\sigma| < \delta} |\sigma| |d\sigma \psi(\sigma, \tau)|.$$

Since $\epsilon > 0$ is arbitrary, property (iii) implies the assertion (52) of Lemma 4.

Proof of (XII). Let $u(x, t)$ be representable in the form (9), in terms of functions (12₁)-(12₃). It will first be shown that if $0 < s < 1$, and if $F'_1(s)$ exists, then

$$(53_1) \quad \lim_{x \rightarrow +0} u_1(x, s) = F'_1(s).$$

The identity (16) shows that

$$-\theta_x(x, t) = \phi(x, t) + \frac{1}{2}\pi^{-1}o(x, t), \quad (t > 0),$$

where $\phi(x, t)$ is the source function (10) and

$$o(x, t) = \sum_{k=1}^{\infty} t^{-3/2} \{ (x + 2k) \exp(-(x + 2k)^2/4t) + (x - 2k) \exp(-(x - 2k)^2/4t) \}.$$

Since this series is uniformly convergent for $0 \leq x \leq x_1 < 2$ and $0 < t \leq 1$ and since each term is 0 for $x = 0$,

$$o(x, t) \rightarrow 0, \text{ as } x \rightarrow 0, \text{ uniformly for } 0 < t \leq 1.$$

Hence, (12₁) shows that (53₁) reduces to

$$\lim_{x \rightarrow +0} \int_0^t \phi(x, t-s) dF_1(s) = F'_1(s),$$

provided that this last relation is valid. To insure its validity, it is sufficient to show that the function $\psi(\sigma, \tau)$ defined by

$$\psi(\sigma, \tau) = \phi(\tau, \sigma) \quad (0 < \tau \leq 1, 0 < \sigma \leq 1)$$

and $\psi(0, \tau) = 0$, possesses properties (i)-(iv), Lemma 4. (In this case, $\psi(\sigma, \tau)$ is defined, not for $|\sigma| \leq l$ but for $0 \leq \sigma \leq 1$. This is irrelevant; in fact, $\psi(\sigma, \tau)$ can be defined to be 0 for negative σ .)

Property (i) is equivalent to $\lim_{x \rightarrow +0} \phi(x, t) = 0$ for $0 < t \leq 1$, which holds, by (10). Property (ii) is equivalent to

$$\lim_{x \rightarrow +0} \int_{\delta}^1 |\phi_t(x, s)| ds = 0 \text{ for } 0 < \delta < 1.$$

Since $2\pi^{\frac{1}{2}}\phi_t(x, t) = -\frac{1}{2}xt^{-5/2}(3 - \frac{1}{2}x^2/t)\exp(-x^2/4t)$, property (ii) holds. Property (iii) is equivalent to

$$\limsup_{x \rightarrow +0} \int_0^{\eta} s |\phi_t(s, x)| ds < \infty \text{ for some } \eta > 0.$$

Since $\partial(t\phi(x, t))/\partial t = \phi(x, t) + t\phi_t(x, t)$, a pair of sufficient conditions for (iii) is

$$(54) \quad \limsup_{x \rightarrow +0} \int_0^{\infty} |d_s s \phi(x, s)| < \infty$$

and

$$(55) \quad \limsup_{x \rightarrow +0} \int_0^{\infty} |\phi(x, s)| ds < \infty.$$

However, by (10), the function $2\pi^{\frac{1}{2}}\partial(t\phi(x, t))/\partial t$, being identical with $\frac{1}{2}xt^{-5/2}(\frac{1}{2}x^2 - t)\exp(-x^2/4t)$, vanishes only for $t = \frac{1}{2}x^2$. Hence, by (10), the integral in (54) is $2(2\pi)^{-\frac{1}{2}}\exp(-\frac{1}{2})$ for all x ; so that (54) holds. The integral in (55) is a constant multiple of

$$\int_0^{\infty} xs^{-3/2} \exp(-x^2/4s) ds = 4 \int_0^{\infty} \exp(-s^2) ds \quad (\frac{1}{2}xs^{-1} \rightarrow s).$$

Hence, (iii) is satisfied. Finally, the integral in (iv) is

$$\int_0^{\delta} s\phi_t(x, s) ds = \delta\phi(x, \delta) - \int_0^{\delta} \phi(x, s) ds.$$

But $\delta\phi(x, \delta) \rightarrow 0$, for $\delta > 0$ fixed, and

$$\int_0^{\delta} \phi(x, s) ds = 2\pi^{-\frac{1}{2}} \int_{\gamma}^{\infty} \exp(-s^2) ds, \quad (\gamma = \frac{1}{2}x\delta^{-1}).$$

Since $\gamma \rightarrow 0$ as $x \rightarrow 0$ for $\delta > 0$ fixed, (iv) is satisfied. This completes the proof of (53₁) for any s on $0 < s < 1$ at which $F_1'(s)$ exists.

Next, it will be shown that

$$(53_2) \quad \lim_{x \rightarrow +0} u_2(x, s) = 0, \quad (0 < s < 1).$$

This is quite on the surface, since $\theta_x(x, t) \rightarrow 0$, as $x \rightarrow 1$, holds uniformly for $0 < t \leq 1$, as can be seen from (16), by writing

$$-2\pi^{\frac{1}{2}}\theta_x(x, t) = \sum_{k=1}^{\infty} t^{-3/2} \{ (x + 2k) \exp(-(x + 2k)^2/4t) \\ + (x - 2k - 2) \exp(-(x - 2k - 2)^2/4t) \}$$

(the latter series is uniformly convergent for $0 < t \leq 1$ and for x on any

closed interval not containing an even integer, while each term is 0 for $x = 1$). Hence, (53₂) follows from (12₂). Also

$$(53_3) \quad \lim_{x \rightarrow +0} u_3(x, s) = 0 \quad (0 < s < 1).$$

This follows from (25), in view of the uniform convergence with respect to x (for $t > 0$ fixed).

The relations (53₁), (53₂), (53₃) imply the statement of (XII) concerning (6₁). Since $\theta(x, t)$ is even, of period 2 in x (for $t > 0$ fixed), the statement concerning (6₂) follows from that concerning (6₁).

In order to prove the assertion of (XII) involving (6₃), it will be shown that

$$(56_1) \quad \lim_{t \rightarrow +0} u_1(x, t) = 0, \quad (0 < x < 1);$$

$$(56_2) \quad \lim_{t \rightarrow +0} u_2(x, t) = 0, \quad (0 < x < 1);$$

and that if $0 < x < 1$ and if the derivative $F_3'(x)$ exists, then

$$(56_3) \quad \lim_{t \rightarrow +0} u_3(x, t) = F_3'(x).$$

The formula (12₁) can be written as

$$u_1(x, t) = - \int_0^t \theta_x(x, s) dF_1(t-s).$$

But (17) implies, for $0 < x < 1$,

$$u_1(x, t) = o(1) \int_0^t |dF_1(s)|, \quad (t \rightarrow 0).$$

Consequently, (56₁) follows from (13₁). Similarly, (56₂) follows from (17) and (13₂). Thus, only (56₃) remains to be proved.

Let $0 < x < 1$ and $0 < \delta < x < 1 - \delta$. Then Lemma 2 and (25) show that, in the proof of (56₃), it can be supposed that $dF_3(s) = 0$ for $0 < s < \delta$ and $1 - \delta < s < 1$; or, more generally, that not only is (13₃) satisfied, but that $F_3(s)$ is defined and of bounded variation $0 \leq s \leq 1$ (and also that $F_3(0) = F_3(1)$). Then, if the identity (16) is used, (12₁) becomes

$$\begin{aligned} \frac{1}{2}(t\pi)^{-\frac{1}{2}} \sum_{k=-\infty}^{+\infty} \left\{ \int_0^1 \exp(-(x-s+2k)^2/4t) dF_3(s) \right. \\ \left. - \int_0^1 \exp(-(x+s+2k)^2/4t) dF_3(s) \right\}. \end{aligned}$$

If $F_3(s)$ is defined for $-\infty < s < \infty$ by the periodicity condition $F_3(s+1)$

$= F_3(s)$, an obvious change of variables in each of the integrals in the last series shows that

$$u_3(x, t) = (t\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \exp(-(x-s)^2/4t) dF_3(s).$$

Hence, (56₃) will follow from Lemma 4 if it is shown that the function

$$\psi(\sigma, \tau) = \frac{1}{2}(\tau\pi)^{-\frac{1}{2}} \exp(-\sigma^2/4\tau)$$

has the properties (i)-(iv), with $l = \infty$.

Condition (i) is implied by $\psi(\pm\infty, \tau) = 0$ and

$$\lim_{\tau \rightarrow +0} \tau^{-\frac{1}{2}} \exp(-\sigma^2/4\tau) = 0, \quad \sigma \neq 0.$$

For a fixed $\tau > 0$, the function $\psi(\sigma, \tau)$ is a monotone function of σ on each of the half-lines $-\infty < \sigma \leq -\delta$, $\delta \leq \sigma < \infty$ and satisfies $\psi(\pm\infty, \tau) = 0$. Hence, (ii) is a consequence of (i). As in the case treated above (in connection with (54) and (55)), a pair of sufficient conditions for (iii) is

$$(57) \quad \limsup_{\tau \rightarrow +0} \int_{-\infty}^{+\infty} \tau^{-\frac{1}{2}} |d\sigma \exp(-\sigma^2/4\tau)| < \infty$$

and

$$\limsup_{\tau \rightarrow +0} \int_{-\infty}^{+\infty} \tau^{-\frac{1}{2}} \exp(-\sigma^2/4\tau) d\sigma < \infty.$$

However, $\partial(\tau^{-\frac{1}{2}} \exp(-\sigma^2/4\tau))/\partial\sigma = \tau^{-\frac{1}{2}}(1 - \frac{1}{2}\sigma^2/\tau) \exp(-\sigma^2/4\tau)$ vanishes only for $\sigma^2 = 2\tau$. Hence, the integral in (57) is $2^{3/2} \exp(-\frac{1}{2})$ for all τ ; so that (57) holds. The change of variables $\sigma/2\tau^{\frac{1}{2}} \rightarrow \sigma$ shows that the integral in the last formula line is the integral of $2 \exp(-\sigma^2)$ over $-\infty < \sigma < \infty$. Thus, condition (iii) is satisfied. Finally, the integral in (iv) is

$$(\tau\pi)^{-\frac{1}{2}} \delta \exp(-\delta^2/4\tau) - \int_{-\delta}^{+\delta} \frac{1}{2}(\tau\pi)^{-\frac{1}{2}} \exp(-\sigma^2/4\tau) d\sigma.$$

For $\delta > 0$ fixed, the first term is $o(1)$ as $\tau \rightarrow +0$; while the change of integration variables $\sigma/2\tau^{\frac{1}{2}} \rightarrow \sigma$ shows that the integral is

$$\pi^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \exp(-\sigma^2) d\sigma + o(1) = 1 + o(1)$$

as $\tau \rightarrow +0$. Thus, (iv) holds.

Consequently, (56₃) follows from Lemma 4. This completes the proof of (XII).

Proof of (V). The existence of the integrals $(8_1)-(8_3)$ as Lebesgue integrals follows from the first part of the proof of (IX); cf. the remarks following the italicized assertion (X). That the sum (9) of $(8_1)-(8_3)$ is a continuous solution of (1) on S is a consequence of (IX). Since (9) is a weighted average of the bounded functions (f_1, f_2, f_3) with non-negative weights and total weight one (cf. the maximum principle used near the end of the proof of (IV)), it follows that (9) is bounded on S . Finally, $(6_1)-(6_3)$ holds almost everywhere by virtue of (XII). This proves (V).

Proof of (VI). It follows from the assertion (X) that, if $u(x, t)$ is a continuous solution of (1) on S and is bounded on S , then u has a representation as a sum (9) of integrals $(12_1)-(12_3)$, where F_1, F_2, F_3 satisfy $(13_1)-(13_3)$, respectively. On the other hand, an examination of the proof of (X) shows that the assumption that u is bounded implies that F_1, F_2, F_3 are absolutely continuous with bounded derivatives. Consequently, u has a representation as a sum (9) of integrals $(8_1)-(8_3)$, where f_1, f_2, f_3 are bounded measurable functions. That u satisfies $(6_1)-(6_3)$ almost everywhere follows from (V). This completes the proof of (VI).

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ON THE DISTRIBUTIONS OF THE ZEROS OF FUNCTIONS BELONGING TO CERTAIN QUASI-ANALYTIC CLASSES.*¹

By I. I. HIRSCHMAN, JR.

1. Introduction. Let $f(t)$ be an infinitely differentiable function defined for $-\infty < t < \infty$, and let M_n be a given sequence of positive constants. We shall write $f(t) \in C\{M_n, k\}$ whenever

$$(1) \quad |f^{(n)}(t)| \leq Ak^n M_n \quad -\infty < t < \infty,$$

for all n and for a suitable constant A which may depend on $f(t)$.

Let $f(t) \in C\{n!, k\}$. Then $f(t)$ is the restriction to the real axis of a function which is analytic in a symmetric strip about the real axis of half width $1/k$ and is bounded in every smaller symmetric strip. Let $Z(u)$ be the number of zeros of $f(t)$ counted according to their multiplicities in the closed interval $-u \leq t \leq u$. It is known that if $f \not\equiv 0$ then

$$(2) \quad \limsup_{t \rightarrow \infty} (2/\pi t) \log Z(t) \leq k.$$

On the other hand if a distribution of zeros is given for which the limit superior in equation (2) is less than k then there exists a function $f(t) \in C\{n!, k\}$ which has precisely these zeros. See [1; vol. II, p. 111].

It is our purpose to extend such theorems to functions belonging to certain quasi-analytic classes. A typical example of our results is that if $f(t) \in C\{n!(\log [n+e])^n, k\}$ and if $Z(u)$ counts the zeros of $f(t)$ as above then

$$(3) \quad \limsup_{t \rightarrow \infty} (2/\pi t) \log \log Z(t) \leq k.$$

On the other hand if a distribution of zeros is given for which the limit superior in equation (3) is less than k , then there exists a function $f(t) \in C\{n!(\log [n+e])^n, k\}$ with precisely these zeros.

An essential step in our procedure is a theorem which we may illustrate by the following particular case.

Let $f(t) \in C\{n!(\log [n+e])^n, k\}$. If for an infinite sequence of intervals

$$|t - t_n| \leq k^{-1} \exp[-\pi k' t_n/2], \quad \lim t_n = +\infty,$$

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we have

$$|f(t)| \leq \exp[-\exp \exp(\pi k' t_n/2)],$$

and if $k' > k$ then $f(t) \equiv 0$. A less precise result was demonstrated in [4] where it was shown that $f \equiv 0$ follows from the assumption

$$|f(t)| \leq \exp[-\exp \exp(\pi k' t/2)] \quad k' > k$$

for all large positive t . The methods of [4] are however applicable to more general quasi-analytic classes than those treated here.

2. Auxiliary functions and preliminary results. Let M_n be a set of positive constants such that $\lim_{n \rightarrow \infty} M_n^{1/n} = \infty$. We set, as is customary,

$$(1) \quad T(v) = \max_{n \geq 0} v^n / M_n \quad 0 \leq v < \infty.$$

We note that

$$(2) \quad \max_{v \geq 0} v^n / T(v) \leq M_n \quad n = 0, 1, \dots$$

See [6]. Given $v \geq 0$ we define $n_T(v)$ to be the largest integer n for which $v^n / M_n = T(v)$. We set

$$(3) \quad H(v) = (2/\pi) \int_1^v \log T(t) t^{-2} dt.$$

It is well known, see [6], that the class $C\{M_n\}$ consisting of all functions $f(t)$ such that for some k , $f(t) \in C\{M_n; k\}$, is quasi-analytic if and only if $H(v) \rightarrow \infty$ as $v \rightarrow \infty$. Thus when $C\{M_n\}$ is quasi-analytic the function $\mathfrak{H}(v)$ which is inverse to $H(v)$ is well defined for $0 \leq v < \infty$, and

$$(4) \quad v = (2/\pi) \int_1^{\mathfrak{H}(v)} \log T(t) t^{-2} dt.$$

We shall need the following results.

LEMMA 2a. Let $M_n = n! [\nu(n)]^n$ where $\nu(0) > 0$, and $\nu(n)$ is non-decreasing. If $f_1(t), f_2(t) \in C\{M_n, h\}$ then $f_1(t) \cdot f_2(t) \in C\{M_n, k\}$ for $k > h$.

We have

$$|f_i^{(n)}(t)| \leq A_i h^n M_n \quad -\infty < t < \infty; i = 1, 2; n = 0, 1, \dots$$

Using Leibnitz's rule we find that

$$[f_1(t)f_2(t)]^{(n)} = \sum_{j=0}^n C_j^n f_1^{(j)}(t) f_2^{(n-j)}(t).$$

$$\begin{aligned} |[f_1(t)f_2(t)]^{(n)}| &\leq A_1 A_2 \sum_{j=0}^n C_j^n h^j j! [\nu(j)]^j h^{n-j} (n-j)! [\nu(n-j)]^{n-j} \\ &\leq A_1 A_2 \sum_{j=0}^n n! [\nu(n)]^j [\nu(n)]^{n-j} h^n \leq A_1 A_2 (n+1) h^n n! [\nu(n)]^n. \end{aligned}$$

Our lemma immediately follows.

By $f(x) \sim g(x)$ and $f(x) \leq \sim g(x)$ we mean respectively that

$$\lim_{x \rightarrow \infty} f(x)/g(x) = 1 \text{ and that } \limsup_{x \rightarrow \infty} f(x)/g(x) \leq 1.$$

LEMMA 2b. Let $M_n = n![v(n)]^n$ where $v(x)$ is continuously differentiable for $0 \leq x < \infty$. If $v(0) = 1$, $v'(x) \geq 0$, $xv'(x)/v(x) = o(1)$ as $x \rightarrow \infty$ then

$$\text{A. } H(ax) \sim H(x) \quad a > 0, x \rightarrow \infty.$$

$$\text{B. } H(x_1 x_2) \leq \sim H(x_1) + H(x_2) \quad x_1, x_2 \rightarrow \infty.$$

$$\text{C. } \theta H(x) \leq \sim H(x^\theta) \quad 0 < \theta < 1, x \rightarrow \infty.$$

$$\text{D. } \log \mathfrak{F}(\lambda x) \geq \sim \lambda \log \mathfrak{F}(x) \quad \lambda > 1, x \rightarrow \infty.$$

By definition

$$\log T(u) = \text{Max}_{n=0,1,\dots} [n \log u - \log \Gamma(n+1) - n \log v(n)].$$

Treating n as a continuous variable and noting that $\log \Gamma(n+1) \sim (n + \frac{1}{2}) \log n - n$; $(d/dn) \log \Gamma(n) = \log n + o(1)$, we have

$$\begin{aligned} (d/dn) [n \log u - \log \Gamma(n+1) - n \log v(n)] \\ = \log u - \log n - \log v(n) - nv'(n)/v(n) + o(1). \end{aligned}$$

This implies that $n_T(u)v[n_T(u)] \sim u$, $\log T(u) \sim n_T(u)$, and, combining these, that $\log T(u) \sim u/v[n_T(u)]$. Noting that v and n_T are both non-decreasing functions, it is apparent that there exists a continuous non-decreasing function $\psi(u)$, for which we may take $\psi(0) > 0$, such that $\log T(u) \sim u/\psi(u)$.

We may now demonstrate conclusion B. The other results may be dealt with in a similar manner. We have

$$\begin{aligned} H(x_1 x_2) &\sim (2/\pi) \int_1^{x_1 x_2} [u\psi(u)]^{-1} du \quad x_1, x_2 \rightarrow \infty \\ &\sim (2/\pi) \int_1^{x_1} [u\psi(u)]^{-1} du + (2/\pi) \int_{x_1}^{x_1 x_2} [u\psi(u)]^{-1} du. \end{aligned}$$

Making the change of variable $x_1 v = u$ in the second integral we have

$$H(x_1 x_2) \sim (2/\pi) \int_1^{x_1} [u\psi(u)]^{-1} du + (2/\pi) \int_1^{x_2} [v\psi(x_1 v)]^{-1} dv,$$

or, since $\psi(u)$ is non-decreasing,

$$\begin{aligned} H(x_1 x_2) &\leq \sim (2/\pi) \int_1^{x_1} [u\psi(u)]^{-1} du + (2/\pi) \int_1^{x_2} [v\psi(v)]^{-1} dv \\ &\leq \sim H(x_1) + H(x_2). \end{aligned}$$

as desired.

We leave to the reader to verify that under the slightly stronger assumption $x\nu'(x)/\nu(x) = 0$ $(\log x)^{-1}$ we have $\log T(u) \sim u/\nu(u)$.

Let us agree to write $\exp_m x$ and $\log_m x$ for

$$\exp \exp \cdots \exp x \text{ and } \log \log \cdots \log x,$$

respectively. For m a positive integer we define

$$N_n^{(m)} = n!, \quad 0 \leq n < \exp_m 1; \quad N_n^{(m)} = n! [\log n \log_2 n \cdots \log_m n]^n, \\ n \geq \exp_m 1.$$

The class $C\{N_n^{(m)}\}$ is called the m -th logarithmic class. It is easily seen from the above remark that for $C\{N_n^{(m)}\}$ we have

$$\log T(v) \sim v/\log v \log_2 v \cdots \log_m v; \quad H(v) \sim (2/\pi) \log_{m+1} v; \\ \log_{m+1} \S(v) \sim (\pi/2)v.$$

See also [4].

Definition 2c. A quasi-analytic class $C\{M_n\}$ will be said to be regular if $M_n = n!\nu(n)^n$ where $\nu(x)$ satisfies the assumptions of Lemma 2b.

We shall make fundamental use of the following theorem due to Gorny [3] and Cartan [2].

THEOREM 2d. *If*

1. $|F^{(n)}(t)| \leq P_n \quad |t - t'| \leq 1/\lambda; n = 0, 1, \cdots,$
2. $P_\tau \geq \tau! P_0 \lambda^\tau,$

then

$$|F^{(k)}(t')| \leq 2e^k P_0^{1-k\tau^{-1}} P_\tau^{k\tau^{-1}} \quad 0 \leq k \leq \tau.$$

3. Functions which are small on a sequence of intervals.

THEOREM 3a. *Let $C\{M_n\}$ be a regular class and let H and \S be defined as in Section 2. If $f(t) \in C\{M_n; k\}$, if for any sequence of pairs $\{t_n, T_n\}$, $0 < t_1 < t_2 < \cdots$, $\lim t_n = \infty$, $0 \leq T_n \leq t_n$, we have in each of the intervals $|t - T_n| \leq 1/k\nu[\S(k't_n)]$ the inequality $|f(t)| \leq O(1)\exp[-\S(k't_n)]$, and if $k' > k$, then $f(t) \equiv 0$.*

We note that it is no restriction to assume that $f(t) \geq 0$, for by Lemma 2a the function $f_1(t) = f(t)^2$ also belongs to $C\{M_n; k_1\}$ for any $k_1 > k$, and $f_1 \equiv 0$ entails $f \equiv 0$.

Next it is no restriction to suppose that $A = 1$, $k = 1$, $k' > 1$, $O(1) = 1$, since we could consider instead of $f(t)$ the function $Bf(tb)$ for suitable constants B and b .

Let us choose constants η_n , $0 \leq \eta_n < 1$, such that $\tau_n = \mathfrak{S}[k't_n] + \eta_n$ is an integer. We apply the theorem of Gorny and Cartan to $f(t)$ with

$$\tau = \tau_n, \quad t' = T_n; \quad \lambda = \nu[\mathfrak{S}(k't_n)] \leq \nu(\tau_n); \\ P_0 = \exp[-\mathfrak{S}(k't_n)] \leq \exp[-\tau_n + 1]; \quad P_\tau = \tau_n! [\nu(\tau_n)]^{\tau_n}.$$

It is immediately verifiable that $P_\tau \geq \tau! P_0 \lambda^\tau$, so that the assumption of Theorem 2d is satisfied. We have

$$|f^{(k)}(T_n)| \leq 6e^{2k} e^{-\tau_n} \{\tau_n! [\nu(\tau_n)]^{\tau_n}\}^{k/\tau_n} \leq 6e^{2k} e^{-\tau_n} [\tau_n \nu(\tau_n)]^k, \\ 0 \leq k \leq \tau_n.$$

Let us define $\phi_n(w) = \int_0^\infty e^{-wt} f(T_n - t) dt$, $w = u + iv$. For $\epsilon > 0$ we

have $|\phi_n(\epsilon \pm iv)| \leq M_0/\epsilon$, $0 \leq v < \infty$. We assert that given θ , $0 < \theta < 1$, the inequality $|\phi_n(\epsilon \pm iv)| \leq 2/\epsilon T(v)$, $1 \leq v \leq \tau_n^\theta$, will hold for all n sufficiently large. We integrate the formula defining $\phi_n(w)$ by parts r times to obtain

$$\phi_n(w) = \left[\sum_{k=0}^{r-1} (-1)^k f^{(k)}(T_n) w^{-k-1} \right] + (-1)^r w^{-r} \int_0^\infty e^{-wt} f^{(r)}(T_n - t) dt.$$

This implies that

$$|\phi_n(\epsilon \pm iv)| \leq \left[\sum_{k=0}^{r-1} |f^{(k)}(T_n)| v^{-k-1} \right] + M_r v^{-r}/\epsilon.$$

It is clearly sufficient to prove that

$$\sum_{k=0}^{n_T(v)-1} |f^{(k)}(T_n)| v^{-k-1} \leq 1/\epsilon T(v) \quad 1 \leq v \leq \tau_n^\theta,$$

where $n_T(v)$ is defined as in Section 2. Now $1/T(v) \geq e^{-v}$, $n_T(v) \leq v + 1$, so that remembering that $v \geq 1$, $T(v) \nearrow$, it is enough to show that

$$\sum_{0 \leq k \leq \tau_n^\theta} 6e^{2k} e^{-\tau_n} [\tau_n \nu(\tau_n)]^k \leq e^{-\tau_n^\theta}/\epsilon$$

for n sufficiently large. Since $\nu(x) = O(x^\delta)$ for any $\delta > 0$ this is obvious.

By the principle of harmonic majoration we now have, for n sufficiently large,

$$\log |\phi_n(1 + \epsilon)| \leq (2/\pi) \left\{ \int_0^1 \log(M_0/\epsilon) [1 + t^2]^{-1} dt \right. \\ \left. + \int_1^{\tau_n^\theta} [\log(2/\epsilon) - \log T(t)] [1 + t^2]^{-1} dt \right. \\ \left. + \int_{\tau_n^\theta}^\infty \log(M_0/\epsilon) [1 + t^2]^{-1} dt \right\}, \leq -H[\tau_n^\theta] + C,$$

where C is a constant independent of n . Using Lemma 2b, and remembering that $\theta < 1$ is arbitrary we see that

$$(1) \quad \log |\phi_n(1 + \epsilon)| \leq \sim -k't_n \quad n \rightarrow \infty.$$

Now $\phi_n(1 + \epsilon) = \int_0^\infty e^{-(1+\epsilon)t} f(T_n - t) dt = e^{-(1+\epsilon)T_n} \int_{-T_n}^\infty e^{-uf}(-u) du$. Since

$f \geq 0$, $\phi_n(1 + \epsilon) \geq e^{-(1+\epsilon)t_n} \int_0^\infty e^{-uf}(-u) du$. Thus

$$(2) \quad \log \phi_n(1 + \epsilon) \geq \sim - (1 + \epsilon)t_n \quad n \rightarrow \infty,$$

if $f \not\equiv 0$. Equations (1) and (2) are in contradiction if ϵ is sufficiently small and this implies that $f(t) \equiv 0$, which is what we wished to prove.

THEOREM 3b. *Let ϵ , $0 < \epsilon \leq 1$, be given; then Theorem 3a remains valid if the inequalities hold in the intervals $|t - T_n| \leq \epsilon\{k_v[\mathfrak{S}(k't_n)]\}^{-1}$ instead of in the intervals $|t - T_n| \leq \{k_v[\mathfrak{S}(k't_n)]\}^{-1}$.*

We shall merely sketch the modifications necessary to prove this slightly more general result. Let $\delta_n = 2\epsilon/k_v[\mathfrak{S}(k't_n)]$, and let N be an integer so large that $(2N + 1)\epsilon \geq 1$. We consider instead of the function $f(u)$ the sequence of functions

$$f_n^*(u) = [f(u)f(u + \delta_n)f(u - \delta_n)f(u + 2\delta_n) \cdots f(u + N\delta_n)f(u - N\delta_n)]^2.$$

If k_1 is any constant $k < k_1 < k'$, then by Lemma 2a, we can choose A_1 independently of n so that

$$|[f_n^*(u)]^{(m)}| \leq A_1 k_1^m M_m, \quad -\infty < u < \infty; m = 0, 1, \dots$$

Further $|f_n^*(u)| \leq O(1)\exp[-\mathfrak{S}(k't_n)]$ for $|u - T_n| \leq 1/k_v[\mathfrak{S}(k't_n)]$. As before we may suppose $k_1 = 1$, etc. We define

$$\phi_n(w) = \int_0^\infty e^{-wv} f_n^*(T_n - u) du.$$

Our assumptions lead, by the arguments used previously, to the relation

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n^*(-u) du = 0,$$

which is impossible unless $f(u) \equiv 0$.

If we apply Theorem 3b, with $\epsilon = 1$ and $T_n = t_n$, to the first logarithmic class, then we obtain the theorem quoted in the introduction. If we apply Theorem 3b to the analytic class and then make an exponential change of variable, we obtain the following result.

COROLLARY 3c. Let $f(z)$ be analytic and bounded in the half plane $\operatorname{Re} z \geq 0$ and let $\epsilon > 0$ be given. If for an infinite sequence of values $0 < x_1 < x_2 < \dots, \lim_{n \rightarrow \infty} x_n = \infty$, we have in each of the intervals

$$(1 - \epsilon)x_n \leq x \leq (1 + \epsilon)x_n$$

the inequalities $|f(x)| \leq e^{-x_n^k}$ and if $k > 1$, then $f(z) \equiv 0$.

This special case can serve as an indication of the degree of precision of our results; see [5; Chapter VII].

4. The zeros of functions belonging to regular classes.

LEMMA 4a. If

$$1. \quad |F^{(n)}(t)| \leq Ak^n M_n \quad -\infty < t < \infty; n = 0, 1, \dots$$

2. $F(t)$ has N zeros, counted according to their multiplicities, in the interval $a \leq t \leq b$, $b - a = 1/\lambda$. Then for every n , $0 \leq n \leq N$, we have $|F(t)| \leq A(k/\lambda)^n M_n/n! \quad a \leq t \leq b$.

Although this is doubtless well known, we shall, for the sake of completeness, give the proof. Let $\xi_1, \xi_2, \dots, \xi_n$ be zeros of $F(t)$ with possible repetitions in the case of multiplicities. Consider a value t , $a \leq t \leq b$. If $F(t) = 0$ there is nothing to prove. If $F(t) \neq 0$ we form the function

$$F(u) \prod_{i=1}^n (t - \xi_i) - F(t) \prod_{i=1}^n (u - \xi_i).$$

This function is zero for $u = t, \xi_1, \xi_2, \dots, \xi_n$, and therefore, by Rolle's theorem, its n -th derivative must vanish at some point $u = \xi$, $a < \xi < b$; i. e.

$$F(t) = F^{(n)}(\xi) [n!]^{-1} \prod_{i=1}^n (t - \xi_i).$$

Since $|t - \xi_i| \leq 1/\lambda$, $|F^{(n)}(\xi)| \leq Ak^n M_n$, we obtain the desired inequality.

THEOREM 4b. Let $C\{M_n\}$ be a regular quasi-analytic class and let $f(t) \in C\{M_n, k\}$, $f(t) \not\equiv 0$. If $Z(t)$ is the number of zeros of $f(u)$ in the interval $-t \leq u \leq t$ counted according to their multiplicities, then

$$(1) \quad \limsup_{t \rightarrow \infty} t^{-1} H[Z(t)] \leq k.$$

Because of conclusion **A** of Lemma 2b it will be enough to show that

$$(2) \quad \limsup_{t \rightarrow \infty} t^{-1} H[Z_+(t)] \leq k$$

where $Z_+(t)$ is the number of zeros of $f(u)$ in the interval $0 \leq u \leq t$. If equation (2) is not true, there exists a constant $k' > k$, and an infinite sequence of values $0 < t_1 < t_2 < \dots$, $\lim t_n = \infty$, such that

$$(3) \quad H(Z_+(t_n)) \geq k't_n \quad n = 1, 2, \dots,$$

or, equivalently, such that

$$(4) \quad Z_+(t_n) \geq \S(k't_n) \quad n = 1, 2, \dots$$

By conclusion **D** of Lemma 2b we see that if we choose k'' , $k < k'' < k'$, then for all sufficiently large n

$$Z_+(t_n) \geq 5kt_n \S(k''t_n) \nu[\S(k''t_n)].$$

We divide the interval $0 \leq t \leq t_n$ into subintervals of length $1/\{4k\nu[\S(k''t_n)]\}$ beginning at t_n and proceeding to the left. There may be a fractional interval abutting zero. There are less than $5kt_n \nu[\S(k''t_n)]$ of these intervals and therefore one of them, which when n is large cannot be the one abutting zero, must contain more than $\S(k''t_n)$ zeros. Let τ_n be the smallest integer greater than or equal to $\S(k''t_n)$ and let T_n be the center of the interval in question. Applying Lemma 4a we find that if

$$(5) \quad |t - T_n| \leq \{8k\nu[\S(k''t_n)]\}^{-1},$$

we have $|f(t)| \leq A\{k\nu(\tau_n)/4k\nu[\S(k''t_n)]\}^{\tau_n}$. But $\lim_{n \rightarrow \infty} \nu(\tau_n)/\nu[\S(k''t_n)] = 1$,

which implies that for n sufficiently large $|f(t)| \leq \exp(-\tau_n)$, or

$$(6) \quad |f(t)| \leq \exp[-\S(k''t_n)].$$

Equations (5) and (6) and Theorem 3b show that $f(t) \equiv 0$ contrary to our assumption. Thus equation (3) must hold and our theorem is proved.

COROLLARY 4c. Let $f(t) \in C\{N_n^{(m)}, k\}$ and let $Z(t)$ be the number of zeros of $f(u)$ in the closed interval from $-u$ to u , counted according to their multiplicities. Then we have $\limsup_{t \rightarrow \infty} (2/\pi t) \log_{m+1} Z(t) \leq k$.

5. On the construction of functions with prescribed zeros. We know from Section 2 that if $C\{M_n\}$ is a regular quasi-analytical class, then $\log T(u) \sim u/\psi(u)$, where $\psi(u)$ is continuous, non-decreasing and $\psi(0) > 0$. We set

$$(1) \quad t = (2/\pi) \int_1^u [u\psi(u)]^{-1} du,$$

$$(2) \quad R(u) = \log \S^*(u), \quad \rho(u) = \psi[\S^*(u)].$$

We have

$$(3) \quad \psi(\exp R(u)) = \rho(u), \quad R(u) = (\pi/2) \int_1^u \rho(x) dx.$$

The first of these relations is immediate; the second may be demonstrated by differentiating equation (1) and the first equation of (2) to obtain $R'(u) = (\pi/2)\rho(u)$, and then noting that $R(0) = 0$.

Let $\epsilon > 0$ be given. It will be convenient to write λ for $1 + \epsilon$.

Let S_z be the strip in the z plane, $z = x + iy$ defined by the inequality $|y| < \pi/2$ for $-\infty < x < \infty$, and let S_w be the curvilinear strip in the w plane, $w = u + iv$ defined by the inequalities $|v| \leq 1/\lambda\rho(0)$ for $-\infty < u \leq 0$, $|v| \leq 1/\lambda\rho(u)$ for $0 \leq u < \infty$. Let $\chi(w)$ be the function which maps S_w onto S_z conformally and is normalized by the conditions $\chi(0) = 0$, $\chi'(0) > 0$.

Let $F(z)$ be analytic and bounded in S_z .

We define

$$(4) \quad f(w) = [4\rho(0)^2 + w^2]^{-1} \exp\{-d(\lambda)\exp[\chi(w)/\lambda]\}F[\chi(w)],$$

where $d(\lambda) = \exp(4\pi/\lambda)/\cos(\pi/2\lambda)$.

THEOREM 5a. *Let $C\{M_n\}$ be a regular quasi-analytic class and let $f(w)$ be defined by equation (4), then $f \in C\{M_n, \lambda^4\}$.*

We shall throughout the present discussion denote by A any constant depending only on $C\{M_n\}$, and the bound of $|F(z)|$ in S_z . Let us define

$$(5) \quad \phi(t) = \int_{-\infty}^{\infty} e^{itw} f(w) dw.$$

We begin by establishing that

$$(6) \quad |\phi(t)| \leq A \exp[-|t|/\lambda\psi(|t|)], \quad -\infty < t < \infty.$$

It is sufficient to consider the case $t \geq 0$. We deform, as we evidently may, the path of integration in the integral (5) until it becomes the upper boundary P of S_w . If P' and P'' are the parts of P in the left and right half planes respectively, then

$$\phi(t) = \int_{P'} e^{itw} f(w) dw + \int_{P''} e^{itw} f(w) dw = I' + I''.$$

Now $|f(w)| \leq A(1 + u^2)^{-1}$ on P' so that $|I'| \leq A \exp(-t/\lambda\rho(0))$ and, a fortiori, $|I'| \leq A \exp[-t/\lambda\psi(t)]$, $t \geq 0$. In order to estimate I'' , we appeal to Ahlfors' distortion theorem, see [1], to show that for w on P'' we have $\operatorname{Re} \chi(w) \geq \lambda R(u) - 4\pi$. This implies that for P on W'' $\exp\{-d(\lambda)\exp[\chi(w)/\lambda]\} \leq \exp[-\exp R(u)]$. If $\sigma(v)$ is the arc length

along P'' between the abscissas $u = 0$ and $u = v$, then $\sigma(u) \leq u + (\lambda\rho(0))^{-1}$, and

$$|I''| \leq A \int_0^\infty \exp[-t/\lambda\rho(u)] \exp[-\exp R(u)] (1+u^2)^{-1} d\sigma(u).$$

Let us put $t = \lambda\rho(\tau) \exp R(\tau)$. We split the above integral into two integrals corresponding to the ranges of integration $(0, \tau)$ and (τ, ∞) . Since $\rho(u)$ and $R(u)$ are non-decreasing we find by simple estimations that $|I''| \leq A \exp[-\exp R(\tau)]$. Thus to prove that $|I''| \leq A \exp[-t/\lambda\psi(t)]$ we need only verify that for τ sufficiently large we have $\exp[-\exp R(\tau)] \leq \exp[-t/\lambda\psi(t)]$, which, inserting for t its expression in terms of τ , will be true if $\psi(\exp R(\tau)) \geq \rho(\tau)$. The validity of this inequality follows from equation (3). Thus equation (6) has been established.

Since $\log T(u) \sim u/\psi(u)$ we have

$$|\phi(t)| \leq A/[T(|t|)]^{1/\lambda^2}, \quad -\infty < t < \infty.$$

By the Fourier inversion formula

$$f(u) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-itu} \phi(t) dt; \quad f^{(n)}(u) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-itu} (-it)^n \phi(t) dt.$$

Hence

$$|f^{(n)}(u)| \leq A \int_0^\infty |t|^n [T(t)]^{-1/\lambda^2} dt.$$

We split this integral into two integrals corresponding to the ranges $(0, 1)$ and $(1, \infty)$. The first of these integrals is $O(1)$ as $n \rightarrow \infty$. Now

$$\int_1^\infty |t|^n [T(t)]^{-1/\lambda^2} dt = \int_1^\infty |t|^{-2} \{ |t|^{(n+2)\lambda^2}/T(t) \}^{1/\lambda^2} dt.$$

Let n_λ be the smallest integer greater than or equal to $(n+2)\lambda^2$. Then by (2) of Section 2 we have

$$\max_{1 \leq t < \infty} t^{(n+2)\lambda^2}/T(t) \leq \max_{0 \leq t < \infty} t^{n_\lambda}/T(t) \leq M_{n_\lambda}.$$

Hence

$$\int_1^\infty |t|^n [T(t)]^{-1/\lambda^2} dt \leq A (n_\lambda!)^{1/\lambda^2} [\nu(n_\lambda)]^{n_\lambda/\lambda^2}.$$

By Stirling's formula we have $(n_\lambda!)^{1/\lambda^2} \leq A\lambda^3 n!$. It may be easily deduced from the relation $x\nu'(x)/\nu(x) = o(1)$ that $\nu(ax) \sim \nu(x)$ for any $a > 0$ so that

$$[\nu(n_\lambda)]^{n_\lambda/\lambda^2} \leq A\lambda^n [\nu(n)]^n.$$

Combining our inequalities we have

$$|f^{(n)}(u)| \leq An! [\nu(n)]^n \lambda^{4n} \quad -\infty < u < \infty; n = 0, 1, \dots,$$

as desired.

THEOREM 5b. *Let $C\{M_n\}$ be a regular quasi-analytic class, let an arbitrary set of real zeros with associated multiplicities be given, and let $Z(u)$ be the number of these zeros counted according to their multiplicities in the closed interval from $-u$ to u . If $\limsup_{t \rightarrow \infty} t^{-1}H[Z(t)] < k$, then there exists a function $f \in C\{M_n, k\}$ which has precisely these zeros.*

Because of Lemma 2a we may restrict ourselves to the case where the zeros are non-negative. It will also be sufficient to suppose $k < 1$ and to prove that $f \in C\{M_n, k'\}$ for any pre-assigned $k' > 1$.

Choose $\epsilon > 0$ so small that if $\lambda = (1 + \epsilon)$ then $\lambda^4 \leq k'$, and let χ, R , etc. be defined as in Theorem 5a. Let $0 \leq \xi_1 \leq \xi_2 \leq \xi_3 \leq \dots$, be the given zeros arranged in order of increasing value and repeated according to their multiplicities. Let $x_k = \chi(\xi_k)$ $n = 1, 2, \dots$, and let $X(u)$ be the number of x_k 's in the closed interval from 0 to u . By Ahlfors' distortion theorem we have $X[\lambda R(u) - 4\pi] \leq Z(u)$. By assumption we have for all u sufficiently large $Z(u) \leq \mathfrak{S}(u)$. Thus if $r = R(u)$ we have for r sufficiently large $X[\lambda r - 4\pi] \leq e^r$, i. e., $X(r) = O(e^{r/\lambda})$, $r \rightarrow \infty$. It follows, see [1], that there exists a function $F(z)$, analytic and bounded in S_z which has the zeros x_k with the correct multiplicities and such that these are the only zeros of $F(z)$.

If now $f(u)$ is defined as in equation (4), then f will have just the prescribed zeros and, by Theorem 5a, $f \in C\{M_n, k'\}$.

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BESSEL FUNCTION APPROXIMATIONS.*

By R. S. PHILLIPS and HENRY MALIN.

In this paper¹ we have obtained the following bounds for the logarithmic derivatives of the modified Bessel and Hankel functions:

$$(1) \quad \begin{aligned} \phi_n\{v, [n(n+1)]^{\frac{1}{2}}\} &< v^{-1}I_n'(v)/I_n(v) < \phi_n(v, n), \\ \phi_n(v, n) &< -v^{-1}K_n'(v)/K_n(v) < \phi_n\{v, [n(n-1)]^{\frac{1}{2}}\} \end{aligned}$$

for $v > 0$ and $n \geq 1$, where

$$(2) \quad \phi_n(v, \alpha) = (n/v^2)[1 + (v/\alpha)^2]^{\frac{1}{2}}.$$

These inequalities result from a study of the behavior of the functions

$$(3) \quad \begin{aligned} X_n(v) &= v^{-1}I_n'(v)/I_n(v) - \phi_n(v, \alpha); \\ Y_n(v) &= -v^{-1}K_n'(v)/K_n(v) - \phi_n(v, \alpha) \end{aligned}$$

by means of the Ricatti differential equation which they satisfy.

THEOREM 1. *If $0 < \alpha \leq n$, then X_n is negative and monotonic increasing, and its graph approaches the v -axis from below; if $n < \alpha < [n(n+1)^2(n+2)]^{\frac{1}{4}}$, then X_n has exactly one maximum and no minima, and its graph thereafter approaches the v -axis from above; if $\alpha \geq [n(n+1)^2(n+2)]^{\frac{1}{4}}$, then X_n is positive and monotonic decreasing, and its graph approaches the v -axis from above.*

This theorem is valid for all integers $n \geq 0$ and $v > 0$.

Making use of the fact that $I_n(v)$ satisfies Bessel's differential equation, it is readily found that X_n satisfies the differential equation

$$(4) \quad \begin{aligned} dZ/dv &= -vZ^2 - 2/v\{n[1 + (v/\alpha)^2]^{\frac{1}{2}} + 1\}Z + v^{-1} \\ &\quad - (n/\alpha)^2v^{-1} - (n/\alpha^2)v^{-1}[1 + (v/\alpha)^2]^{-\frac{1}{2}}. \end{aligned}$$

One can likewise write down the series and asymptotic expansions for $X_n(v)$ from known expansions for $I_n(v)$; they are:

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$$(5) \quad X_n(v) = \{[2(n+1)]^{-1} - n/(2\alpha^2)\} \\ - \{[8(n+1)^2(n+2)]^{-1} - n/(8\alpha^4)\}v^2 + \dots,$$

$$(6) \quad X_n(v) = [1 - (n/\alpha)]v^{-1} - (1/2)v^{-2} + \dots$$

The differential equation (4) associates a direction to every point in the (v, Z) -plane. The point and associated direction are known as a line-element. In particular we will be interested in the locus of all points where the slope of the line element is zero. Setting dZ/dv equal to zero results in a quadratic equation in Z . The roots of this quadratic in Z are two functions of v . Let $U(v)$ be the greater and $L(v)$ be the smaller of these two roots. The curves representing these two functions divide the half-plane $v \geq 0$ into three regions: the region above the U -curve in which dZ/dv is negative; the region between the U - and L -curves in which dZ/dv is positive; and the region below the L -curve in which dZ/dv is again negative.

Most of the proof is concerned with an investigation of the U - and L -curves. The behavior of the X_n curve for small v is known by the series (5). We know therefore to which of the three regions X_n belongs initially. Recall that X_n belongs to the family of Z -curves. It follows that X_n will be monotonic increasing (decreasing) until it intersects with zero slope, a U - or L -curve, after which it is monotonic decreasing (increasing); etc. The known behavior of the X_n at infinity enables us to complete the theorem. It turns out that the X_n behave like $U = U(v; \alpha, n)$.

We begin, then, by setting dZ/dv equal to zero in equation (4) and find that the explicit expressions for U and L are

$$(7) \quad U, L = v^{-1}[-v^{-1}(ns + 1) \\ \pm \{1 + (n^2 + 1)v^{-2} + (n/s)(\alpha^{-2} + 2v^{-2})\}^{\frac{1}{2}}],$$

where $s = [1 + (v/\alpha)^2]^{\frac{1}{2}}$; the plus sign goes with U , and the minus sign with L . It is clear from equation (7) that U and L are always real valued so that their graphs actually do divide the right half-plane into the three regions described above.

One sees from equation (7) that $L(v)$ is the sum of negative monotonic increasing terms and hence is itself negative monotonic increasing. Furthermore, $L(v)$ has a pole at the origin. It follows from (5) that the graph of X_n starts out above that of $L(v)$. Furthermore if it should ever touch the $L(v)$ -curve it would cross (having a zero slope at this point) and enter a region of negative slope. It could never again intersect the $L(v)$ -curve since $L(v)$ is a monotonic increasing function. The value of X_n at this point of intersection would thereafter be an upper bound for X_n . Since this value

is necessarily less than zero, this would be contrary to the fact that X_n approaches zero asymptotically as $v \rightarrow \infty$. Consequently the graph of X_n lies above that of $L(v)$ for all $v \geq 0$.

The quadratic equation in Z obtained from equation (4) by setting dZ/dv equal to zero can be rewritten as a cubic function of s .² The result is

$$(8) \quad F(s) \equiv \alpha^2 Z^2 s^3 + 2nZs^2 - [\alpha^2 Z^2 + 1 - 2Z - n^2/\alpha^2]s + n/\alpha^2 = 0.$$

To a given value of Z and a root $s \geq 1$, there is a $v \geq 0$ such that either $U(v) = Z$ or $L(v) = Z$. (This value of Z need not of course be assumed by the function X_n .) Since $F(-\infty) = -\infty$, $F(0) = n/\alpha^2$, and $F(\infty) = \infty$, it follows for a given Z , $F(s)$ has at most two roots for which $s \geq 1$. Since $L(v)$ takes on all negative values, it follows that for $Z < 0$, one of these roots and only one necessarily corresponds to a point on $L(v)$. Hence in the region where $U(v)$ is negative, it must be monotonic increasing. On the other hand, for positive Z , either zero, one, or two roots can lie on $U(v)$. Hence in the region where $U(v)$ is positive it is either monotonic or has a single maximum (since, as is readily shown, the graph of $U(v)$ approaches the v -axis asymptotically as $v \rightarrow \infty$).

To conclude our description of $U(v)$ we make use of the series and asymptotic expansions of $U(v)$ which are readily obtainable from equation (7).

$$(9) \quad U(v) = \{[2(n+1)]^{-1} - n/(2\alpha^2)\} \\ - \{[8(n+1)^3]^{-1} - (n(n+2))/[8(n+1)\alpha^4]\}v^2 \\ + \{[16(n+1)^5]^{-1} - n/[16(n+1)^3\alpha^4]\} \\ - [n(n+3)]/[16(n+1)\alpha^6]\}v^4 + \dots,$$

$$(10) \quad U(v) = [1 - (n/\alpha)]v^{-1} - [1 - n/(2\alpha)]v^{-2} + \dots$$

For $\alpha \leq n$, $U(v)$ is negative for v near zero and also for sufficiently large v ; thus $U(v)$ is always negative and monotonic increasing. For $n < \alpha < [n(n+1)^2(n+2)]^{1/4}$, $U(v)$ has a positive slope for small v and is positive as $v \rightarrow \infty$, which implies that it has a single maximum and no minima. Finally for $\alpha \geq [n(n+1)^2(n+2)]^{1/4}$, $U(v)$ has a negative slope for small values of v and is positive as $v \rightarrow \infty$, and therefore is always positive and monotonic decreasing.

² The change of variable $s^2 = 1 + (v/\alpha)^2$ establishes a 1-1 relationship between the two half-planes $s \geq 1$ and $v \geq 0$. The strip $0 \leq s < 1$ does not, of course, correspond to any part of the real v -plane. Curves in the s -half-plane go over into curves in the v -half-plane under a horizontal stretching process; the property of being monotonic and the property of having a horizontal tangent are preserved by the transformation. It is for this reason that we shall translate freely these properties proved in the s -variable into the v -variable without further comment.

We now return to the function X_n . A comparison of the series expansion for X_n and U [equations (5) and (9)] shows that for $\alpha \leq [n(n+1)^2(n+2)]^{1/4}$ the graph of X_n starts below that of $U(v)$. Now the graph of X_n cannot intersect that of $U(v)$ from below at a point at which $U(v)$ is increasing, because at such a point the slope of X_n would be zero (by definition of U). For $\alpha \leq n$, U is always monotonic increasing so that X_n remains between L and U , that is in the positive slope region. Consequently in this case X_n is negative and monotonic increasing.

For $n < \alpha < [n(n+1)^2(n+2)]^{1/4}$, X_n can intersect U only at a point at which U has a non-positive slope. At such a point it must necessarily cross to the region above U , since U is either decreasing or at its maximum, whereas X_n has a zero slope and can only decrease by crossing over to the region above U . Further, it is clear that X_n must cross into this region since it approaches the v -axis from above and hence eventually is decreasing. Once in the region above U , X_n must remain there for all larger v , since U is thereafter a monotonic decreasing function and X_n would necessarily have a zero slope at any point of intersection. Thus X_n is increasing at the start, has its maximum at the point of intersection with U and is thereafter decreasing, approaching the v -axis asymptotically from above.

Finally, for $\alpha > [n(n+1)^2(n+2)]^{1/4}$, X_n is greater than U at the start and since U is monotonic decreasing for $v > 0$, it follows, as above, that X_n must remain above U for all $v > 0$. Hence in this case X_n is positive and monotonic decreasing. This is likewise true when the inequality is replaced by an equality since the functions X_n are continuous in α . This concludes the proof of Theorem 1.

Upper and lower bounds of the functions $v^{-1}I_n'(v)/I_n(v)$ and $[v^{-1}I_n'(v)/I_n(v)]'$ can now be readily deduced. If $\alpha^2 = n(n+1)$, the function $X_n = 0$ for $v = 0$ and is thereafter positive; for $\alpha = n$, X_n is always negative. If $\alpha = [n(n+1)^2(n+2)]^{1/4}$, then X_n' is negative for $v > 0$; whereas if $\alpha = n$, X_n' is positive for $v > 0$. These results are summarized in terms of $\phi_n(v, \alpha)$ [see (2)] by means of the following corollary:

COROLLARY 1.1. For all $v > 0$ and $n \geq 1$,

$$\begin{aligned}\phi_n\{v, [n(n+1)]^{1/4}\} &< v^{-1}I_n'(v)/I_n(v) < \phi_n(v, n), \\ \phi_n'(v, n) &< [v^{-1}I_n'(v)/I_n(v)]' < \phi_n'\{v, [n(n+1)^2(n+2)]^{1/4}\}.\end{aligned}$$

We next proceed to prove a theorem similar to Theorem 1 for the function $Y_n(v)$ related as in (3) to the modified Hankel function $K_n(v)$.

THEOREM 2. If $\alpha \leq [n(n-1)^2(n-2)]^{1/4}$, then Y_n is negative and

monotonic increasing, and its graph approaches the v -axis from below; if $[n(n-1)^2(n-2)]^{1/4} < \alpha < n$, then Y_n has a single minimum and no maxima and its graph thereafter approaches the v -axis from below; if $\alpha \geq n$, then Y_n is positive and monotonic decreasing, and its graph approaches the v -axis from above.

This theorem is valid for all $v > 0$ and $n \geq 0$.

The series and asymptotic expansions for Y_n are:

$$\begin{aligned} Y_0 &= -[v^2 \log(\gamma v/2)]^{-1} + \dots, \\ (11) \quad Y_1 &= -\log(\gamma v/2) - (2\alpha^2)^{-1} \dots, \\ Y_n &= \{[2(n-1)]^{-1} - n/(2\alpha^2)\} \\ &\quad - \{[8(n-1)^2(n-2)]^{-1} - n/\alpha^4\}v^2 + \dots \end{aligned}$$

for $n > 1$, where $C = \log \gamma = 0.577 \dots$ is Euler's constant.

$$(12) \quad Y_n = [1 - n/\alpha]v^{-1} + (1/2)v^{-2} + \dots$$

for $n \geq 0$ and large v . Further $Y_n(v)$ satisfies the following differential equation:

$$\begin{aligned} (13) \quad dZ/dv &= vZ^2 + (2/v)\{n[1 + (v/\alpha)^2]^{-1} - 1\}Z - v^{-1} \\ &\quad + (n/\alpha)^2v^{-1} - (n/\alpha^2)v^{-1}[1 + (v/\alpha)^2]^{-1}. \end{aligned}$$

We shall first show that $Y_0(v)$ is a positive monotonic decreasing function of v for all $v > 0$. It follows from (11) that Y_0 is positive for small v ; it is further analytic in v for $v > 0$. If Y_0 were ever to become negative, there would exist a $v = v_1$ such that $Y_0(v_1) = 0$. For such a $v = v_1$, equation (13) shows that dY_0/dv becomes $-v_1^{-1} < 0$. Hence, although Y_0 could presumably become zero and take on negative values, it could never thereafter become positive. However, by the asymptotic expansion (12), Y_0 approaches the v -axis from above as $v \rightarrow \infty$. This behavior is consistent only with the assumption that $Y_0 > 0$ for all $v > 0$.

To go on to the monotonicity of Y_0 , we see from equation (11) that dY_0/dv is negative for v small. Differentiating equation (13) we get ($n=0$)

$$(14) \quad d^2Y_0/dv^2 = [2vY_0 - (2/v)]dY_0/dv + Y_0^2 + (2/v^2)Y_0 + v^{-2}.$$

For any $v = v_1$ at which $dY_0/dv = 0$, equations (13) and (14) show that d^2Y_0/dv^2 reduces to $2[Y_0(v_1)]^2 > 0$. Hence if dY_0/dv were ever to vanish, it must change from negative to positive values. That is to say if dY_0/dv ever becomes positive, it must remain so. This is inconsistent with the asymptotic behavior of Y_0 as $v \rightarrow \infty$. Hence dY_0/dv is negative for all $v > 0$.

We next consider the general case $n \geq 1$. As in Theorem 1, we are interested in the locus of points where $dZ/dv = 0$. This locus is obtained as the solution of a quadratic in Z and defines the two functions $U(v)$ and $L(v)$ which are respectively the greater and the smaller of the roots of this quadratic. From equation (13), the explicit expressions for U and L are:

$$(15) \quad U, L = v^2(1 - ns) \pm \{1 + (1 + n^2)v^2 - (n/s)(\alpha^2 + 2v^2)\}^{\frac{1}{2}},$$

where $s = [1 + (v/\alpha)^2]^{\frac{1}{2}}$; the plus sign goes with U and the minus sign with L .

It is no longer true that U and L are defined for all v and n . One can therefore not expect U and L to have the simple properties that they had in Theorem 1; it is precisely this fact that complicates the proof of the present theorem. If U and L are everywhere defined, the half-plane $v > 0$ breaks up into three regions in each of which the sign of dZ/dv is everywhere the same: the region above the U -curve in which dZ/dv is positive, the region between the U - and L -curves in which dZ/dv is negative, and the region below the L -curve in which dZ/dv is positive. When U and L are not everywhere defined, the U - and L -curves have two branches; in the regions interior to the U -, L -curves, dZ/dv is negative; in the region exterior to the U -, L -curves, dZ/dv is positive.

One can readily obtain the series and asymptotic expansions of U from equation (15). They are

$$(16) \quad \begin{aligned} U_1 &= v^{-1} - (2\alpha^2)^{-1} + \dots, \\ U_n &= \{[2(n-1)]^{-1} - n/(2\alpha^2)\} \\ &\quad - [8(n-1)]^{-1}\{(n-1)^{-2} - n(n-2)/\alpha^4\}v^2 + \dots \quad (n > 1) \end{aligned}$$

for small v , and

$$(17) \quad U_n = [1 - (n/\alpha)]v^{-1} + [1 - n/(2\alpha)]v^{-2} + \dots, \quad (n \geq 1),$$

for large v .

We see from equation (15) that for those values of v for which L exists, L is the sum of two negative terms and is therefore itself negative. Furthermore, L has a pole at the origin. It follows that any solution curve which is bounded from below at the origin (hence Y_n in particular) starts out above the L -curve.

As in Theorem 1, we shall again study the U - and L -curves by means of the cubic function of s obtained from equation (13) by setting $dZ/dv = 0$, namely,

$$(18) \quad F(s) \equiv \alpha^2 Z^2 s^3 + 2nZs^2 - \{\alpha^2 Z^2 + 2Z + 1 - (n^2/\alpha^2)\}s - n/\alpha^2 = 0.$$

To each value of Z and a root $s \geq 1$ of $F(s) = 0$, there is a v such that either $U(v) = Z$ or $L(v) = Z$.

We shall first consider the case $\alpha \geq n$. It is easy to see that the radicand in equation (15) is positive since $1 > (sn)^{-1} \geq n/(s\alpha^2)$ and $1 + n^2 \geq 2n > 2n/s$ for $s > 1$. It follows that U and L are defined (and single valued) for all $s > 1$. Furthermore, for $Z > 0$ and $\alpha \geq n$ we see that the coefficients of $F(s)$ have the signs $(+ + - -)$ in the order of decreasing powers of s . Thus there is just one variation in sign, and by Descartes' rule of signs there is precisely one root for $s > 0$. Now the U -curve has a positive ordinate and negative slope at $s = 1$ and approaches the v -axis from above as $v \rightarrow \infty$. If U were to have a minimum, there would be values of Z such that $U(v) = Z$ for at least two different values of v , and hence two different positive values of s . Thus $F(s) = 0$ for more than one positive value of s . Since this is impossible, U can have no minimum and we conclude that U is positive and monotonic decreasing for $\alpha \geq n$.

From the expansion of Y_n [equation (11)] we see that the graph of Y_n starts off positive with a negative slope and hence lies between the U -curve and the L -curve for small $v > 0$. Since U is monotonic decreasing, the graph of Y_n could cross that of U with zero slope. If it did so, however, it would thereafter be in the region of positive slope above the graph of U and hence would be bounded away from zero. This is contrary to the fact that Y_n approaches zero from above as v approaches infinity [see equation (12)]. Finally if Y_n were ever zero, the asymptotic behavior of Y_n indicates that it would be zero an even number of times. This means that, at the points of crossing, its slope would alternate in sign. This is impossible since for $Y_n = Z = 0$, equation (13) reduces to

$$dZ/dv = -v^{-1}[1 - (n/\alpha)^2] - (n/\alpha^2)v^{-1}[1 + (v/\alpha)^2]^{-1} < 0 \text{ for all } v > 0.$$

It follows that Y_n is positive and monotonic decreasing for $\alpha \geq n$.

We have as a corollary to the first part of the theorem that

$$(19) \quad -v^{-1}K'_n(v)/K_n(v) > (n/v^2)[1 + (v/n)^2]^{-1/2}$$

for all $v > 0$. Hence if we define the auxiliary function

$$(20) \quad \psi(v) = (n/v^2)\{[1 + (v/n)^2]^{-1/2} - [1 + (v/\alpha)^2]^{-1/2}\}$$

it follows from (19) that, for all $v > 0$ and all $\alpha > 0$,

$$(21) \quad Y_n(v) > \psi(v).$$

We now consider the case $\alpha < n$. In this case U and L need not be

defined for all $v \geq 0$. It will be convenient to consider U and L as functions³ of s ; that is, as roots of $F(s) = 0$.

$$U, L = [\alpha^2(s^2 - 1)]^{-1} \{1 - ns \pm [1 + n^2 + \alpha^2(s^2 - 1) - (n/s)(1 + s^2)]^{1/2}\}.$$

For s small and positive, the radicand is negative so that the curves do not exist in this region. Each curve has its only positive singularity at $s = 1$. This can be seen from the expansion of U about $s = 1$

$$U = (1 - n)/[\alpha^2(s - 1)] + \dots \quad (s < 1, n > 1)$$

$$U = \alpha^{-1}[2(s - 1)]^{-1/2} + \dots \quad (s > 1, n = 1).$$

Now for $Z > 0$ the coefficients of $F(s)$ have the signs $(+ + \pm -)$ and hence $F(s)$ has one and only one positive real root. If $Z = 0$, $F(s) = 0$ for $s = [n(1 - \sigma)]^{-1}$, where we have set $\sigma = (\alpha/n)^2$. For any $Z \geq 0$ the value of s satisfying $F(s) = 0$ makes $U(s) = Z$. Hence, the graph of $Z = U(s)$ crosses the s -axis at $P_1: ([n(1 - \sigma)]^{-1}, 0)$ and crosses every line $Z = \text{constant} > 0$ just once. If we try to trace the curve $Z = U(s)$ we find first that for $n > 1$ it approaches the line $s = 1$ asymptotically from the left; it cannot cross the Z -axis, so that it must double back on itself; since it crosses each horizontal line ($Z > 0$) just once it can have no minimum; $F(s)$ is quadratic in Z and hence the curve cannot double back on itself more than once for $s < 1$; for $s > 1$ it cannot double back on itself above the s -axis, since this would imply a root $L > 0$, which is impossible for $s > 1$ (i.e. $v > 0$). It follows that for $Z > 0$ and $n > 1$ the curve must look like the sketches in Fig. a and b.

If $n = 1$, the graph of $Z = U$ approaches the line $s = 1$ asymptotically from the right, and since it crosses the s -axis at $(1 - \sigma)^{-1} > 1$, it must decrease monotonically until it reaches this point. (See Fig. c; $Z > 0$).

We shall study U for $Z < 0$ by means of the auxiliary function $\psi(v)$ defined by equation (20). At $v = 0$, $U = [2(n - 1)]^{-1} - n/(2\alpha^2)$, $\psi = (2n)^{-1} - n/(2\alpha^2)$, and $L \rightarrow -\infty$ as $v \rightarrow 0+$; therefore ψ lies between U and L at $v = 0$. In order to determine the points at which the graph of ψ intersects that of U or L we go back to the equation defining U and L ; that is, we insert $Z = \psi(v)$ in equation (13), setting $dZ/dv = 0$. One obtains by direct substitution of (20) into (13)

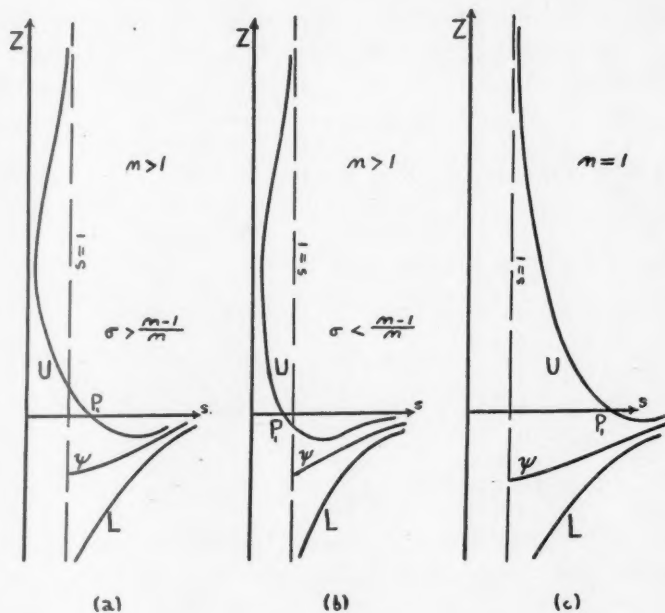
$$(2n/v^3) \{ [1 + (v/\alpha)^2]^{1/2} - [1 + (v/n)^2]^{1/2} \} - [n/(v\alpha^2)] [1 + (v/\alpha)^2]^{-1/2} = 0.$$

Solving for the real roots in v (that is $s \geq 1$), we get $(v/\alpha)^2 = 4\sigma/(1 - 4\sigma)$, ($\sigma = (\alpha/n)^2 > 0$), if and only if $\sigma < 1/4$. For this value of v ,

³ It should be noted that for $0 < s < 1$, L can be positive; whereas for $1 \leq s$ (i.e. $v > 0$) L must be negative where it exists.

$$\psi = -[n/(2\alpha^2)][1 - 4\sigma]^{\frac{1}{2}} \text{ and } s = [1 - 4\sigma]^{-\frac{1}{2}}.$$

It follows that for $n/2 \leq \alpha < n$, the graph of ψ intersects neither the U - nor the L -curves for $v \geq 0$, since in this case $\sigma \geq \frac{1}{4}$. This implies that U and L are not connected and hence must exist and be single-valued for all $v > 0$. The graph of L starts at minus infinity and approaches the v -axis asymptotically from below. Hence L must assume all negative values of Z at least once. For a given value of $Z < 0$, U can therefore correspond to at



most two roots of the cubic $F(s) = 0$. Since the U -curve crosses into the region $Z < 0$ from above and approaches the s -axis asymptotically from below, U can have but a single minimum for α in the range $n/2 \leq \alpha < n$. It is clear that $U > \psi > L$. The situation is therefore represented by the sketches in Fig. a, b, c.⁴

For $\alpha < n/2$, the function ψ has just one point in common with U or L ($v \geq 0$), namely P_2 :

$$([1 - 4\sigma]^{-\frac{1}{2}}, -(2\sigma n)^{-1}[1 - 4\sigma]^{\frac{1}{2}})$$

⁴ It should be noted that for $s \geq 1$, U is single-valued because of the existence of L below it. For $s < 1$, however, the extension of U to the left of $s = 1$ (the lower part of which would properly be called L) is double-valued because of the existence of the upper branch of U which approaches $s = 1$ asymptotically from the left (if $n > 1$).

(in the (sZ) -plane). Now

$$\psi - (U + L)/2 = \{n[1 + (v/n)^2]^{\frac{1}{2}} - 1\}v^{-2} > 0;$$

thus ψ is always greater than the average of U and L ($v \geq 0$), and hence the graph of ψ certainly lies above that of L . Therefore it intersects the graph of U once ($v \geq 0$) and thereafter lies above both the U - and the L -curves.

We now wish to show that to the left of P_2 , that is, for $0 < s \leq [1 - 4\sigma]^{-\frac{1}{2}}$, U has at most one minimum and no maximum. We show first that in this range, $F(s)$ has precisely one root for $-(2n\sigma)^{-1}[1 - 4\sigma]^{\frac{1}{2}} < Z < 0$. It follows that there is one value of s for each Z in this range for which $U(s) = Z$. We define $G(Z)$ as follows:

$$G(Z) \equiv [(4\sigma^2 n^2)(1 - 4\sigma)^{-3/2}]Z^2 + 2[n - (1 - 4\sigma)^{\frac{1}{2}}][1 - 4\sigma]^{-1}Z \\ + [n(1 - \sigma) - (1 - 4\sigma)^{\frac{1}{2}}][n\sigma(1 - 4\sigma)^{\frac{1}{2}}]^{-1} = F[(1 - 4\sigma)^{-\frac{1}{2}}].$$

$G(Z)$ is thus a parabolic function of Z , and

$$G(0) = [n(1 - \sigma) - (1 - 4\sigma)^{\frac{1}{2}}][n\sigma(1 - 4\sigma)^{\frac{1}{2}}]^{-1} > 0,$$

since $[1 - 4\sigma]^{\frac{1}{2}}/n \leq [1 - 4\sigma]^{\frac{1}{2}} < 1 - 2\sigma < 1 - \sigma$. Further since P_2 lies on U , $G(-[1 - 4\sigma]^{\frac{1}{2}}/(2n\sigma)) = 0$. The minimum of the parabolic function, $G(Z)$, is

$$Z_m = -[1 - 4\sigma]^{\frac{1}{2}}/(2n\sigma) \cdot [n - (1 - 4\sigma)^{\frac{1}{2}}]/(2n\sigma) < -[1 - 4\sigma]^{\frac{1}{2}}/(2n\sigma),$$

since as above $(n - [1 - 4\sigma]^{\frac{1}{2}}) > 2n\sigma$. Hence Z_m lies below P_2 . Now $G(Z) = 0$ for $Z = -[1 - 4\sigma]^{\frac{1}{2}}/(2n\sigma)$ and the minimum of $G(Z)$ occurs for $Z < -[1 - 4\sigma]^{\frac{1}{2}}/(2n\sigma)$; it follows that $G(Z)$ is greater than 0 for all Z in the range $-[1 - 4\sigma]^{\frac{1}{2}}/(2n\sigma) < Z \leq 0$. Returning to the function $F(s)$, we have

$$F(0) = -n^2/\alpha^2 < 0 \quad (Z \text{ arbitrary});$$

$$F([1 - 4\sigma]^{-\frac{1}{2}}) \equiv G(Z) > 0 \quad (\text{for } -[1 - 4\sigma]^{\frac{1}{2}}/(2n\sigma) < Z \leq 0).$$

Hence the equation $F(s) = 0$ has an odd number of roots for $0 \leq s \leq [1 - 4\sigma]^{-\frac{1}{2}}$ and fixed Z , ($-[1 - 4\sigma]^{\frac{1}{2}}/(2n\sigma) < Z \leq 0$). If we can show that the graph of $F(s)$ has no point of inflection in this region it will follow that there is exactly one root. Now $F''(s) = 0$ for $s = -2/(3n\sigma Z)$. For $Z > -[1 - 4\sigma]^{\frac{1}{2}}/(2n\sigma)$, i. e. $-Z < [1 - 4\sigma]^{\frac{1}{2}}/(2n\sigma)$, this value of $s > (4/3)(1 - 4\sigma)^{-\frac{1}{2}} > (1 - 4\sigma)^{-\frac{1}{2}}$; thus the point of inflection is to the right of P_2 .

As we have seen, the U -curve enters the lower half of the (s, Z) -plane by crossing the s -axis at the point P_1 . Since $F(s)$ has precisely one root to the left of P_2 for each negative Z above P_2 , the U -curve must proceed downward without a maximum or a minimum in the region R above and to the left of P_2 . It must pass out of R at P_2 or to the left of P_2 , since P_2 would otherwise lie on L . If it passes out of R to the left of P_2 it can not again enter R , because if it were to do so $F(s)$ would have more than one root for some value of Z in R . On the other hand, the curve eventually passes through P_2 , so that it must have a minimum below P_2 . Furthermore it can have no other extremum below P_2 , since $F(s)$ can have at most three roots for any value of Z . Finally for v in the range $0 \leq v \leq 2\alpha[\sigma/(1-4\sigma)]^{\frac{1}{2}}$ (i.e. $1 \leq s \leq [1-4\sigma]^{-\frac{1}{2}}$), the curve L exists below ψ ; it follows that $U(v)$ is single-valued in this range. Thus $U(v)$ is single-valued and has at most one minimum but no maxima to the left of P_2 . There are thus only three possible descriptions of $U(v)$ to the left of P_2 . $U(v)$ is monotonic decreasing; $U(v)$ has a single minimum and no maximum; $U(v)$ is monotonic increasing. The latter case occurs only if $U(v)$ is initially increasing, i.e. if $\alpha \leq [n(n-1)^2(n-2)]^{\frac{1}{4}}$.

The proof of the theorem for $\alpha < n$ is now straightforward. When $[n(n-1)^2(n-2)]^{\frac{1}{4}} < \alpha < n$ the graph of Y_n starts off below the U -curve with a negative slope. In the upper half-plane, U decreases monotonically until its graph intersects the v -axis. The graph of Y cannot intersect that of U above the v -axis, for if it did so it would enter a region of positive slope and remain bounded away from the v -axis; this is contrary to the behavior of Y_n at infinity. Furthermore, the graph of Y_n cannot intersect (with zero slope) that of U where U is increasing. On the other hand, since Y_n is eventually increasing, it must enter a region of positive slope at some time. Now $Y_n > \psi > L$. Hence the graph of Y_n intersects the U -curve below the v -axis at a point at which U is decreasing. Thereafter Y_n cannot enter a region of negative slope. For if it intersected the graph of U again, the intersection would have to occur at a point of increasing slope for U (to the left of P_2 if $\alpha < n/2$). In this case Y_n would become trapped between ψ and a monotonically increasing portion of U ; it could never again enter a region of increasing slope. This is contrary to the behavior of Y_n at infinity.

For $\alpha \leq [n(n-1)^2(n-2)]^{\frac{1}{4}}$, the graphs of both Y_n and U start out with positive slopes and with Y_n above U . In this case U is monotonic increasing at least up to the point P_2 . Hence if Y_n ever entered a region of decreasing slope, it would have to intersect the U -curve to the left of P_2 .

As before Y_n would thereafter be trapped between ψ and a monotonically increasing portion of U . In this case Y_n would remain in a region of decreasing slope and therefore be bounded away from the v -axis; this is contrary to the behavior of Y_n at infinity. It follows that Y_n is monotonic increasing for all $v > 0$. This concludes the proof of Theorem 2.

Theorem 2 furnishes us with some useful bounds on the function $-v^{-1}K'_n(v)/K_n(v)$ and its derivative. For $\alpha^2 = n(n-1)$, Y_n is initially zero and is negative thereafter; for $\alpha = n$, Y_n is always positive. For $\alpha = [n(n-1)^2(n-2)]^{1/4}$, dY_n/dv is positive ($v > 0$); for $\alpha = n$, dY_n/dv is negative ($v > 0$). We have thus proved

COROLLARY 2.1. For all $n \geq 1$ and all $v > 0$,

$$\begin{aligned} \phi_n(v, n) &< -v^{-1}K'_n(v)/K_n(v) < \phi_n\{v, [n(n-1)]^{1/2}\}, \\ \phi'_n\{v, [n(n-1)^2(n-2)]^{1/4}\} &< [-v^{-1}K'_n(v)/K_n(v)]' < \phi'_n(v, n). \end{aligned}$$

Combining the results of the previous corollaries to Theorems 1 and 2,

$$-v^{-1}K'_n(v)/K_n(v) > \phi_n(v, n) > v^{-1}I'_n(v)/I_n(v).$$

In other words, $I'_n(v)/I_n(v) + K'_n(v)/K_n(v) < 0$, and hence:

COROLLARY 2.2. For all $v \geq 0$ and $n \geq 0$,

$$[I_n(v)K_n(v)]' < 0.$$

The case $n = 0$ requires a special and somewhat tedious argument. For the details of the proof we refer the reader to the original report of which this paper is a part.⁵

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⁵ See pp. 54-57 of 'A Helical Wave Guide' by Phillips and Malin, New York University Mathematical Research group, report No. 170-3, Army Air Forces, Watson Laboratories.

ON THE APPROXIMATION OF IRRATIONAL NUMBERS BY THE CONVERGENTS OF THEIR CONTINUED FRACTIONS, II.*

By ALFRED BRAUER and NATHANIEL MACON.

Introduction. This paper is a continuation of our previous paper with the same title [this JOURNAL, vol. 71 (1949), pp. 349-361]. The enumeration of the chapters, theorems, and equations will be continued here. We are now able to improve the results of Chapter 4, in which we obtained lower bounds for the sums of two, three, four, and eight consecutive λ_n and used the results to obtain a lower bound for $\lim_{m \rightarrow \infty} (\sum_{i=0}^{m-1} \lambda_i)/m$. In this paper, these results are not only improved, but in addition we obtain the best possible bound for the sum of k consecutive λ_n for any given k , and the best possible bound for $\lim_{m \rightarrow \infty} (\sum_{i=0}^{m-1} \lambda_i)/m$.

If

$$(45) \quad f_{-2} = 1, f_{-1} = 0, f_0 = 1, f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 5, \dots$$

are the numbers of Fibonacci, then we set

$$(46) \quad F_k = \sum_{i=0}^{k-1} f_{i-1}/f_i.$$

It will be shown that for every irrational number ξ , for every k , and for every n , we have $\sum_{\kappa=0}^{k-1} \lambda_{n+\kappa} > k + 2F_k$, but for every integer k and for every $\epsilon > 0$ it is possible to construct irrational numbers ξ for which $\sum_{\kappa=0}^{k-1} \lambda_{n+\kappa} < k + 2F_k + \epsilon$, for infinitely many values of n .

A theorem of Hurwitz [10] can be formulated as follows:

$$\lim_{m \rightarrow \infty} (\sum_{i=0}^{m-1} \lambda_i)/m = 5^{1/2}$$

for $\xi = (1 + 5^{1/2})/2$ and equivalent numbers ξ .

The following generalization of this theorem will be proved. For every irrational number ξ , we have $\lim_{m \rightarrow \infty} (\sum_{i=0}^{m-1} \lambda_i)/m \geq 5^{1/2}$.

* Received May 2, 1949. Presented to the American Mathematical Society, December 29, 1949.

5. **Some lemmas.** Let $\xi = [q_0, q_1, \dots, q_n, \dots]$ be a given irrational number. We shall consider the sum $\lambda_n + \lambda_{n+1} + \dots + \lambda_{n+k-1}$ for any given n and any $k > 1$.

Let t be an integer such that $1 \leq t \leq k$, and let $q_{n+t} + 1 = q_{n+t}^*$. Consider the irrational number $\xi^* = [q_0, q_1, \dots, q_{n+t-1}, q_{n+t}^*, q_{n+t+1}, \dots]$, and denote by λ_{n+k}^* , ξ_{n+k}^* , ϕ_{n+k}^* the values corresponding to λ_{n+k} , ξ_{n+k} , ϕ_{n+k} , respectively, for ξ^* .

LEMMA 2. We have

$$(47) \quad \sum_{\kappa=1}^k (\xi_{n+\kappa}^* - \xi_{n+\kappa}) > \frac{1}{2}.$$

Proof. It follows from (6) that

$$(48) \quad \xi_{n+\kappa} = [q_{n+\kappa}, q_{n+\kappa+1}, \dots].$$

Hence $\xi_{n+\kappa}^* = \xi_{n+\kappa}$, ($\kappa = t+1, t+2, \dots$), and it is sufficient to show that

$$(49) \quad \sum_{\kappa=t}^1 (\xi_{n+\kappa}^* - \xi_{n+\kappa}) > \frac{1}{2}.$$

Now

$$(50) \quad \xi_{n+t}^* - \xi_{n+t} = 1,$$

and

$$\begin{aligned} \xi_{n+t-1}^* - \xi_{n+t-1} &= q_{n+t-1} + 1/(\xi_{n+t} + 1) - q_{n+t-1} - 1/\xi_{n+t} \\ &= -1/\xi_{n+t}(\xi_{n+t} + 1). \end{aligned}$$

Since $\xi_{n+t} > 1$, we have

$$(51) \quad -\frac{1}{2} < \xi_{n+t-1}^* - \xi_{n+t-1} < 0.$$

Moreover, $\xi_{n+\kappa-1}^* - \xi_{n+\kappa-1} = 1/\xi_{n+\kappa}^* - 1/\xi_{n+\kappa} = (\xi_{n+\kappa} - \xi_{n+\kappa}^*) / (\xi_{n+\kappa} \xi_{n+\kappa}^*)$, ($\kappa = t-1, t-2, \dots, 1$). Hence the left member of (49) is an alternating series with decreasing terms. Thus it follows from (50) and (51)

that $\sum_{\kappa=t}^1 (\xi_{n+\kappa}^* - \xi_{n+\kappa}) > 1 - \frac{1}{2} = \frac{1}{2}$.

LEMMA 3. We have

$$(52) \quad \sum_{\kappa=1}^k (\phi_{n+\kappa}^* - \phi_{n+\kappa}) > -\frac{1}{2}.$$

Proof. It follows from (7) that $\phi_{n+\kappa} = [0, q_{n+\kappa-1}, q_{n+\kappa-2}, \dots, q_1]$. Hence, $\phi_{n+\kappa}^* = \phi_{n+\kappa}$, ($\kappa = 1, 2, \dots, t$), and we only need to show that

$$(53) \quad \sum_{\kappa=t+1}^k (\phi_{n+\kappa}^* - \phi_{n+\kappa}) > -\frac{1}{2}.$$

Now $\phi_{n+t+1}^* - \phi_{n+t+1} = 1/(q_{n+t} + 1 + \phi_{n+t}) - 1/(q_{n+t} + \phi_{n+t}) = -1/(q_{n+t} + 1 + \phi_{n+t})(q_{n+t} + \phi_{n+t})$. Hence

$$(54) \quad \phi_{n+t+1}^* - \phi_{n+t+1} > -\frac{1}{2}.$$

In addition, for $\kappa = t+1, t+2, \dots$,

$$\begin{aligned} \phi_{n+\kappa}^* - \phi_{n+\kappa} &= (\phi_{n+\kappa}^* + q_{n+\kappa}) - (\phi_{n+\kappa} + q_{n+\kappa}) = 1/\phi_{n+\kappa+1}^* - 1/\phi_{n+\kappa+1} \\ &= (\phi_{n+\kappa+1} - \phi_{n+\kappa+1}^*)/\phi_{n+\kappa+1}\phi_{n+\kappa+1}^*. \end{aligned}$$

Since $0 < \phi_{n+\kappa+1}\phi_{n+\kappa+1}^* < 1$, the left member of (53) is an alternating series with decreasing terms. Thus (53) follows from (54).

LEMMA 4. For every $k > 1$, we have $\sum_{\kappa=0}^{k-1} (\lambda_{n+\kappa}^* - \lambda_{n+\kappa}) > 0$.

Proof. By Lemmas 2 and 3, we have $\sum_{\kappa=0}^{k-1} (\lambda_{n+\kappa}^* - \lambda_{n+\kappa}) = \sum_{\kappa=1}^k (\xi_{n+\kappa}^* - \xi_{n+\kappa}) + \sum_{\kappa=1}^k (\phi_{n+\kappa}^* - \phi_{n+\kappa}) > \frac{1}{2} - \frac{1}{2} = 0$. By successive application of Lemma 4, we obtain

LEMMA 5. Let $\xi = [q_0, q_1, \dots, q_n, q_{n+2}, \dots, q_{n+k}, q_{n+k+1}, \dots]$ be an arbitrary irrational number. If we set

$$(55) \quad \bar{\xi} = [q_0, q_1, \dots, q_n, 1, 1, \dots, 1, q_{n+k+1}, q_{n+k+2}, \dots],$$

then $\sum_{\kappa=0}^{k-1} \lambda_{n+\kappa} \geq \sum_{\kappa=0}^{k-1} \bar{\lambda}_{n+\kappa}$ where the $\bar{\lambda}_{n+\kappa}$ are the λ -values associated with $\bar{\xi}$.

LEMMA 6. Let x be the finite continued fraction $[q_0, q_1, \dots, q_n, 1, 1, \dots, 1]$ where the last k partial quotients are each unity. If the complete quotients are denoted by x_v , then $\sum_{\kappa=1}^k x_{n+\kappa} = k + F_k$, where F_k is defined by (45) and (46).

Proof. It follows from the recursion formula for the convergents that

$$(56) \quad x_{n+k-i} = f_{i+1}/f_i, \quad (i = 0, 1, \dots, k-1).$$

$$\text{Hence } \sum_{\kappa=1}^k x_{n+\kappa} = \sum_{i=0}^{k-1} f_{i+1}/f_i = k + \sum_{i=0}^{k-1} f_{i-1}/f_i = k + F_k.$$

LEMMA 7. Let $\bar{\xi}$ again denote the continued fraction (55). Then the complete quotients satisfy the inequality $\sum_{\kappa=1}^k \bar{\xi}_{n+\kappa} > k + F_k$.

Proof. From (48) we obtain here $\bar{\xi}_{n+k-i} = [1, 1, \dots, 1, \bar{\xi}_{n+k+1}]$, ($i = 0$,

$1, \dots, k-1$), where the first $i+1$ partial quotients are 1's. It follows from the recursion formula for the continued fraction and from (56) that

$$(57) \quad \bar{\xi}_{n+k-i} = (f_{i+1}\bar{\xi}_{n+k+1} + f_i)/(f_i\bar{\xi}_{n+k+1} + f_{i-1}), \quad (i = 0, 1, \dots, k-1).$$

Differentiating with respect to $\bar{\xi}_{n+k+1}$, we have $d\bar{\xi}_{n+k-i}/d\bar{\xi}_{n+k+1} = (f_i f_{i+1} \bar{\xi}_{n+k+1} + f_{i-1} f_{i+1} - f_i f_{i+1} \bar{\xi}_{n+k+1} - f_i^2)/(f_i \bar{\xi}_{n+k+1} + f_{i-1})^2$. Using the formula $f_{i-1} f_{i+1} - f_i^2 = (-1)^{i+1}$ for the Fibonacci numbers, we obtain

$$d\bar{\xi}_{n+k-i}/d\bar{\xi}_{n+k+1} = (-1)^{i+1}/(f_i \bar{\xi}_{n+k+1} + f_{i-1})^2.$$

Hence $\sum_{i=0}^{k-1} (d\bar{\xi}_{n+k-i}/d\bar{\xi}_{n+k+1}) = \sum_{i=0}^{k-1} (-1)^{i+1}/(f_i \bar{\xi}_{n+k+1} + f_{i-1})^2$. The right member is a finite alternating series with decreasing terms. Since the first term is negative, the sum is negative. Hence $\sum_{\kappa=1}^k \bar{\xi}_{n+\kappa}$ is a decreasing function of $\bar{\xi}_{n+k+1}$. Therefore, by (57),

$$\begin{aligned} \sum_{\kappa=1}^k \bar{\xi}_{n+\kappa} &> \lim_{\bar{\xi}_{n+k+1} \rightarrow \infty} \sum_{i=0}^{k-1} (f_{i+1} \bar{\xi}_{n+k+1} + f_i)/(f_i \bar{\xi}_{n+k+1} + f_{i-1}) \\ &= \sum_{i=0}^{k-1} f_{i+1}/f_i = \sum_{i=0}^{k-1} (1 + f_{i-1}/f_i) = k + F_k. \end{aligned}$$

Corresponding to Lemma 7, we have

LEMMA 8. $\sum_{\kappa=1}^k \bar{\phi}_{n+\kappa} \geq F_k$, the equality holding only if $n = 0$.

Proof. It follows from (7) that

$$(58) \quad \bar{\phi}_{n+\kappa} = [0, q_{n+\kappa-1}, q_{n+\kappa-2}, \dots, q_1].$$

If $n = 0$, then proceeding as in the proof of Lemma 6, we obtain $\bar{\phi}_{n+\kappa} = \bar{\phi}_\kappa = [0, 1, 1, \dots, 1] = f_{\kappa-2}/f_{\kappa-1}$, ($\kappa = 1, 2, \dots, k$). Hence $\sum_{\kappa=1}^k \bar{\phi}_\kappa = \sum_{\kappa=1}^k f_{\kappa-2}/f_{\kappa-1} = \sum_{\kappa=0}^{k-1} f_{\kappa-1}/f_\kappa = F_k$.

Now consider the case $n \geq 1$. We obtain from (7) $\bar{\phi}_{n+\kappa} = [0, 1, 1, \dots, 1, q_n + \bar{\phi}_n]$, where the 1's occur $\kappa-1$ times. Similarly to (57), we get

$$(59) \quad \bar{\phi}_{n+\kappa} = [f_{\kappa-2}(q_n + \bar{\phi}_n) + f_{\kappa-3}]/[f_{\kappa-1}(q_n + \bar{\phi}_n) + f_{\kappa-2}].$$

Upon differentiating with respect to $q_n + \bar{\phi}_n$ and summing, we obtain

$$\sum_{\kappa=1}^k d\bar{\phi}_{n+\kappa}/d(q_n + \bar{\phi}_n) = \sum_{\kappa=1}^k (-1)^\kappa/[f_{\kappa-1}(q_n + \bar{\phi}_n) + f_{\kappa-2}]^2.$$

It follows as in Lemma 7 that $\sum_{\kappa=1}^k \bar{\phi}_{n+\kappa}$ is a decreasing function of $q_n + \bar{\phi}_n$.

Therefore, by (59),

$$\sum_{\kappa=1}^k \bar{\phi}_{n+\kappa} > \lim_{\kappa=1}^k [f_{\kappa-2}(q_n + \bar{\phi}_n) + f_{\kappa-3}] / [f_{\kappa-1}(q_n + \bar{\phi}_n) + f_{\kappa-2}],$$

where $q_n + \bar{\phi}_n \rightarrow \infty$; i. e.,

$$\sum_{\kappa=1}^k \bar{\phi}_{n+\kappa} > \sum_{\kappa=0}^{k-1} f_{\kappa-1} / f_{\kappa} = F_k.$$

6. A generalization of a theorem of Hurwitz. We are now able to prove the following theorem:

THEOREM 17. *For every irrational number ξ , for every $k > 0$, and for every n , we have $\sum_{\kappa=0}^{k-1} \lambda_{n+\kappa} > k + 2F_k$.*

Proof. For $k = 1$, the theorem is trivial. For given n and $k > 1$, we define $\bar{\xi}$ as in (55). It follows from Lemma 5 that

$$(60) \quad \sum_{\kappa=0}^{k-1} \lambda_{n+\kappa} \geq \sum_{\kappa=0}^{k-1} \bar{\lambda}_{n+\kappa},$$

and from Lemmas 7 and 8 that

$$(61) \quad \sum_{\kappa=0}^{k-1} \bar{\lambda}_{n+\kappa} = \sum_{\kappa=1}^k (\bar{\xi}_{n+\kappa} + \phi_{n+\kappa}) > k + 2F_k.$$

The theorem follows from (60) and (61).

Theorem 17 contains Theorems 4, 12, 13 and 15 as special cases. More exactly, it follows that the sum of any three consecutive λ_n is greater than 6 for every irrational number ξ . For the sums of four, five, six, seven and eight consecutive λ_n , we obtain correspondingly 25/3, 158/15, 767/60, 11711/780, and 31399/1820. Using this last bound for eight consecutive λ_n , we obtain $\lim_{m \rightarrow \infty} (\sum_{i=0}^{m-1} \lambda_i) / m > 2.1565$ instead of Theorem 16. However, this result can be improved as follows:

THEOREM 18. *For every irrational number ξ , we have*

$$\lim_{m \rightarrow \infty} (\sum_{i=0}^{m-1} \lambda_i) / m \geq 5^{\frac{1}{2}}.$$

Proof. From Theorem 17, we have $\sum_{i=0}^{m-1} \lambda_i / m > (m + 2F_m) / m$. Now it is well known that the quotient of two consecutive Fibonacci numbers f_i / f_{i-1} , tends to $(5^{\frac{1}{2}} + 1) / 2$ as i tends to infinity. Thus it follows from (46) that $\lim_{m \rightarrow \infty} F_m / m = 2 / (5^{\frac{1}{2}} + 1)$. Hence, $\lim_{m \rightarrow \infty} (\sum_{i=0}^{m-1} \lambda_i) / m \geq 1 + 4 / (5^{\frac{1}{2}} + 1) = 5^{\frac{1}{2}}$.

The example $\xi = (5^{\frac{1}{2}} + 1)/2$ shows that Theorem 18 cannot be improved. In addition, we wish to prove that the bound given in Theorem 17 is the best possible.

THEOREM 19. *For every given integer k and for every $\epsilon > 0$ it is possible to find irrational numbers ξ such that, for infinitely many values of n ,*

$$\sum_{\kappa=0}^{k-1} \lambda_{n+\kappa} < k + 2F_k + \epsilon.$$

Proof. Let n be an arbitrary positive integer, and choose any number $\bar{\xi}$ for which $q_{n+\kappa} = 1$, ($\kappa = 1, 2, \dots, k$). Then it follows from (57) that

$$\sum_{\kappa=1}^k \bar{\xi}_{n+\kappa} = \sum_{i=0}^{k-1} \bar{\xi}_{n+k-i} = \sum_{i=0}^{k-1} (f_{i+1} \bar{\xi}_{n+k+1} + f_i) / (f_i \bar{\xi}_{n+k+1} + f_{i-1}).$$

It was shown in the proof of Lemma 7 that this sum is a decreasing function of $\bar{\xi}_{n+k+1}$ and that the right member tends to $k + F_k$ as $\bar{\xi}_{n+k+1}$ becomes infinite, i. e. as q_{n+k+1} becomes infinite. Thus, for sufficiently large q_{n+k+1} , we have

$$(62) \quad \sum_{\kappa=1}^k \bar{\xi}_{n+\kappa} < k + F_k + \frac{1}{2}\epsilon.$$

Similarly, it follows from the proof of Lemma 8 that $\lim_{q_n \rightarrow \infty} \sum_{\kappa=1}^k \bar{\phi}_{n+\kappa} < k + F_k + \frac{1}{2}\epsilon$. Hence, for sufficiently large q_n , we have

$$(63) \quad \sum_{\kappa=1}^k \bar{\phi}_{n+\kappa} < F_k + \frac{1}{2}\epsilon.$$

If we choose an irrational number ξ such that $q_{n+1} = q_{n+2} = \dots = q_{n+k} = 1$ and such that q_n and q_{n+k+1} are so large that (62) and (63) are satisfied, then $\sum_{\kappa=0}^{k-1} \lambda_{n+\kappa} < k + 2F_k + \epsilon$ for this number ξ .

If $n_1 < n_2 < \dots$ is a set of positive integers such that $n_{i+1} - n_i \geq k + 2$, ($i = 1, 2, \dots$), then in the same way we can construct irrational numbers ξ such that

$$\sum_{\kappa=0}^{k-1} \lambda_{n_i+\kappa} < k + 2F_k + \epsilon, \quad (i = 1, 2, \dots).$$

Thus the proof is complete.

We use the opportunity to remark that in the statement of Theorem 11, " $n > 2$ " should be inserted after $\lambda_{n-2}, \lambda_{n-1}, \dots, \lambda_{n+3m-1}$.

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